Topology Comprehensive Exam
Spring 2019

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let X be a path-connected space. Show that $\pi_1(X, x_0)$ is abelian if and only if all basepoint-changing homomorphisms β_γ depend only on the endpoints of the path γ. Recall that for a path γ from x_0 to x_1, the homomorphism $\beta_\gamma \colon \pi_1(X, x_1) \to \pi_1(X, x_0)$ is defined to be $\beta_\gamma([f]) = [\gamma \cdot f \cdot \overline{\gamma}]$.

2. Let M be a compact oriented n–manifold without boundary. Let η be a $(n-1)$–form on M. Show that there is some point on M such that the n–form $d\eta$ is zero

3. Use algebraic topology to prove that $\mathbf{Z}_2 * \mathbf{Z}$, i.e., the free product of $\mathbf{Z}_2$ and $\mathbf{Z}$, has two subgroups of index two that are not isomorphic to one another.

4. Suppose that M and N are smooth manifolds of dimension m and n, respectively, and S a submanifold of N of codimension greater than $m+1$. Given functions $f_i; M \to N, i = 0, 1$ such that $f_i(M) \cap S = \emptyset$, show that f_0 and f_1 are smoothly homotopic as maps to N if and only if they are smoothly homotopic as maps to $N - S$.

5. Let $M(n, \mathbf{R})$ be the set of all n by n matrices (recall that it can be identified with $\mathbf{R}^{n^2}$ by choosing an ordering of the entries of the matrix). Let $SL(n, \mathbf{R})$ be the set of matrices with determinant 1. Show that $SL(n, \mathbf{R})$ is a manifold and compute its dimension. What is the tangent space to $SL(N, \mathbf{R})$ at the identity matrix?
   Hint: You may use the fact that the derivative of the determinate map at A applied to a tangent vector B is given by $d(det)_A(B) = det(A)tr(A^{-1}B)$.

6. Recall that the suspension SX of a space X is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. Let X be any space (not necessarily path-connected). Show that $S(S(X))$ is simply-connected.

7. Let M and N be two smooth manifolds and $f : M \to N$ a smooth function. If M is compact and f is an immersion, then shown that $f^{-1}(x)$ is finite for any $x \in N$.

8. Let M and N be closed oriented manifolds of dimensions m and n, respectively. Show that the de Rham cohomology space $H^n_{DR}(M \times N)$ is non-zero.