Topology Comprehensive Exam
Spring 2019

Student Number: 

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1  2  3  4  5  6  7  8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.
1. Let $X$ be a path-connected space. Show that $\pi_1(X, x_0)$ is abelian if and only if all basepoint-changing homomorphisms $\beta_\gamma$ depend only on the endpoints of the path $\gamma$. Recall that for a path $\gamma$ from $x_0$ to $x_1$, the homomorphism $\beta_\gamma: \pi_1(X, x_1) \to \pi_1(X, x_0)$ is defined to be $\beta_\gamma([f]) = [\gamma \cdot f \cdot \bar{\gamma}]$.

2. Let $M$ be a compact oriented $n$–manifold without boundary. Let $\eta$ be a $(n-1)$–form on $M$. Show that there is some point on $M$ such that the $n$–form $d\eta$ is zero.

3. Use algebraic topology to prove that $\mathbb{Z}_2 * \mathbb{Z}$, i.e., the free product of $\mathbb{Z}_2$ and $\mathbb{Z}$, has two subgroups of index two that are not isomorphic to one another.

4. Suppose that $M$ and $N$ are smooth manifolds of dimension $m$ and $n$, respectively, and $S$ a submanifold of $N$ of codimension greater than $m+1$. Given functions $f_i: M \to N$, $i = 0, 1$ such that $f_i(M) \cap S = \emptyset$, show that $f_0$ and $f_1$ are smoothly homotopic as maps to $N$ if and only if they are smoothly homotopic as maps to $N - S$.

5. Let $M(n, \mathbb{R})$ be the set of all $n$ by $n$ matrices (recall that it can be identified with $\mathbb{R}^{n^2}$ by choosing an ordering of the entries of the matrix). Let $SL(n, \mathbb{R})$ be the set of matrices with determinant 1. Show that $SL(n, \mathbb{R})$ is a manifold and compute its dimension. What is the tangent space to $SL(N, \mathbb{R})$ at the identity matrix? **Hint:** You may use the fact that the derivative of the determinate map at $A$ applied to a tangent vector $B$ is given by $d(det)A(B) = det(A)tr(A^{-1}B)$.

6. Recall that the suspension $SX$ of a space $X$ is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. Let $X$ be any space (not necessarily path-connected). Show that $S(S(X))$ is simply-connected.

7. Let $M$ and $N$ be two smooth manifolds and $f : M \to N$ a smooth function. If $M$ is compact and $f$ is an immersion, then shown that $f^{-1}(x)$ is finite for any $x \in N$.

8. Let $M$ and $N$ be closed oriented manifolds of dimensions $m$ and $n$, respectively. Show that the de Rham cohomology space $H^n_{DR}(M \times N)$ is non-zero.