Topology Comprehensive Exam
Fall 2018

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Suppose that $\pi : \widetilde{X} \to X$ is a covering space for which $\pi^{-1}(x)$ is countable for some $x \in X$. If X has the structure of a smooth manifold, then show that $\widetilde{X}$ can be given the structure of a smooth manifold so that π is an immersion. (Be sure to check that $\widetilde{X}$ satisfies all the properties necessary to be a manifold.)

2. Let M and W be compact m and $(n+m)$ dimensional oriented manifolds, respectively. Suppose that S is a compact oriented submanifold of W of dimension n. (The manifolds M, W, and S are all without boundary.) Let $f : M \to W$ be a smooth function that is transverse to S. Show that the intersection number $I(f, S)$ of f and S vanishes in the following two cases
    1. M is the boundary of a compact manifold Y and f extends over Y.
    2. S is the boundary of a compact submanifold of W.

3. Let G be a topological group. Recall that means that G is a topological space and a group such that the maps $\mu : G \times G \to G : (g, h) \mapsto g \cdot h$ and $i : G \to G : g \mapsto g^{-1}$ are continuous (here $g \cdot h$ is group multiplication). Given two loops $\gamma, \eta : [0,1] \to G$ based at the identity element e in G we denote the concatenation of the paths by $\gamma * \eta$ and we define the path $\gamma \cdot \eta$ to be
$$\gamma \cdot \eta : [0,1] \to G : t \mapsto \gamma(t) \cdot \eta(t).$$
Prove the following:
    1. $\gamma * \eta$ is homotopic to $\gamma \cdot \eta$ by a homotopy that is fixed at the end points.
    2. the fundamental group $\pi_1(G, e)$ is abelian.

4. Thinking of the projective plan $\mathbf{R}P^2$ as $\mathbf{R}^3 - \{(0,0,0)\}$ modulo the equivalence relation $(x, y, z) \sim (tx, ty, tz)$ for some non-zero t. The function $\widetilde{f}(x, y, z) = \frac{x^2+2y^2}{x^2+y^2+z^2}$ descends to a well-defined function $f : \mathbf{R}P^2 \to \mathbf{R}$. Prove the function is smooth and find its critical points. Show that one of the critical points is non-degenerate. (Actually all are non-degenerate, that is f is a Morse function, but the computation would take too long. So just verify that one of the critical points you found is non-degenerate.)

5. Write down an explicit closed 1–form on $\mathbf{R}^2 - \{(0,0)\}$ that is not exact. Carefully prove the form has both these properties.

6. Let X and Y be two non-empty spaces. If X is path connected and Y has two path components then show that the join $X * Y$ is simply-connected (this is true without the hypothesis on Y, but you only need to prove it in the stated case). Recall the join is the quotient space of $X \times Y \times [0,1]$ where each of the sets $X \times \{y\} \times \{1\}$, for $y \in Y$, and $\{x\} \times Y \times \{0\}$, for $x \in X$, is collapsed to a separate point.

7. Let $f : X \to Y$ and $g : Y \to Z$ be covering spaces with Z connected and $g^{-1}(z)$ finite for some $z \in Z$. Show that $g \circ f : X \to Z$ is a covering space. What goes wrong with the proof if $g^{-1}(z)$ is infinite?

8. Suppose that M and N are both smooth oriented compact n–manifolds (without boundary) and $f : M \to N$ is a smooth map. If there is an n-form η on N such that $\int_M f^*\eta \neq 0$ then show that f is surjective.