Algebra Comprehensive Exam
January 15, 2016

Student Number: ☐

Instructions: Complete up to 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. List all intermediate fields K such that $\mathbb{Q} \subset K \subset F$, and find all elements $\alpha \in F$ such that $F = \mathbb{Q}(\alpha)$.

2. Show that two commuting complex square matrices share an eigenvector, without using the result that they are simultaneously triangularizable.

3. Show that all groups of order 35 are cyclic.

4. Let G be a finite group, and let H be a proper subgroup of G. Prove that the union of all conjugates of H is a proper subset of G. Show that the conclusion need not be true if G is infinite.

5. Is the ring $\mathbb{Z}[2i]$, where $(2i)^2 = -4$, a principal ideal domain? If not, give an example of a non-principal ideal.

6. An R-module M is called faithful if $rM = 0$ for $r \in R$ implies $r = 0$. Let M be a finitely generated faithful R-module and let J be an ideal of R such that $JM = M$. Prove that $J = R$.

7. Let R be an integral domain. Show that every automorphism of $R[x]$ that is identity on R is given by $x \mapsto ax + b$ where $a, b \in R$ and a is a unit.

8. Let p be a prime and $q = p^n$ for some positive integer n. Show that the map $x \mapsto x^p$ is an automorphism on $\mathbb{F}_q$ to itself. Describe all automorphisms on $\mathbb{F}_q$.