Algebra Comprehensive Exam
Spring 2018

Student Number: 

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1 2 3 4 5 6 7 8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.
1. Let $H$ be the subgroup of $S_6$ generated by $(16425)$ and $(16)(25)(34)$. Let $H$ act on $S_6$ by conjugation. Show that the set


is invariant under $H$, thereby defining a homomorphism $\phi : H \to S_5$. Show that $\phi$ is an isomorphism.

2. Show that every finite group is isomorphic to a subgroup of a simple group.

3. Let $R$ be a commutative ring with 1. Suppose an ideal $I$ in $R$ is such that $xy \in I$ implies that either $x \in I$ or $y^n \in I$. Let

$$\sqrt{I} = \{r \in R : r^n \text{ for some } n \in \mathbb{Z}_{>0}\}$$

Show that $\sqrt{I}$ is the smallest prime ideal containing $I$. (Here “smallest” means that any other prime ideal containing $I$, contains $\sqrt{I}$. Hint: remember to prove that $\sqrt{I}$ is an ideal, which is prime.)

4. Suppose that $R$ is a commutative ring with 1 such that for every $x \in R$, there is some natural number $n > 1$ such that $x^n = x$. Show that every prime ideal of $R$ is maximal.

5. Compute the Galois group of $x^4 - x^2 - 6$ over $\mathbb{Q}$.

6. Suppose $V$ is a finite dimensional vector space over a field $k$ and suppose that $A : V \to V$ is a $k$-linear endomorphism whose minimal polynomial is not equal to its characteristic polynomial. Show that there exist $k$-linear endomorphisms $B, C : V \to V$ such that $AB = BA$, $AC = CA$, but $BC \neq CB$.

7. Suppose that $K$ is an extension of $\mathbb{Q}$ of degree $n$. Let $\sigma_1, \ldots, \sigma_n : K \hookrightarrow \mathbb{C}$ be the distinct embeddings of $K$ into $\mathbb{C}$. Let $\alpha \in K$. Regarding $K$ as a vector space over $\mathbb{Q}$, let $\phi : K \to K$ be the linear transformation $\phi(x) = \alpha x$. Show that the eigenvalues of $\phi$ are $\sigma_1(\alpha), \ldots, \sigma_n(\alpha)$.

8. An $R$-module $M$ is called irreducible if $M \neq 0$ and the only submodules of $M$ are 0 and $M$. Now suppose that $R$ is a commutative ring with 1 and that $M$ is a left $R$-module. Show that $M$ is irreducible if and only if $M$ is isomorphic to $R/I$ for a maximal ideal $I$ of $R$. 