Analysis Comprehensive Exam
January 22, 2016

Student Number: ☐

Instructions: Complete up to 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

NOTES:

- All functions in this exam are (extended) real-valued.
- The exterior Lebesgue measure of $E \subseteq \mathbb{R}^d$ is denoted by $|E|_e$, and if E is measurable then its Lebesgue measure is $|E|$.
- The characteristic function of a set A is denoted by χ_A.
- A consequence of the Stone–Weierstrass theorem is the Weierstrass Approximation Theorem, which states that the set of polynomials on $[0,1]^d$ is dense in $C([0,1]^d)$ if the latter set is endowed with the uniform norm. You can use this fact without proof.
- You can also use without proof the fact the every monotone function $f : [a,b] \to \mathbb{R}$ is Borel measurable and is differentiable almost everywhere.

1. Prove that $E \subseteq \mathbb{R}^d$ is measurable if and only if $|Q| = |Q \cap E|_e + |Q \setminus E|_e$. for every box Q.

2. Let μ be a positive, Borel regular measure on $I = [0,1]$ such that $\mu(I) = 1$. Set $\xi_n(x) = x^n$ for $n = 0, 1, 2, \ldots$ and $x \in I$. Let

$$H = L^2(I, \mu), \quad V = L^\infty(I, \mu).$$

Let $\langle \cdot, \cdot \rangle_H$ denote the inner product on H and $|| \cdot ||_H$ the norm on H. Let $|| \cdot ||_V$ denote the norm on V.

(i) Show that $||\xi_n||_H \to 0$ if and only if $\mu\{1\} = 0$.

(ii) Show that if $f \in H$ and $\langle f, \xi_n \rangle_H = 0$ for all $n = 0, 1, 2, \ldots$, then $f = 0$ μ-almost everywhere.

(iii) Show that $||\xi_n||_V \to 0$ if and only if for some $\epsilon > 0$ we have $\mu([1-\epsilon, 1]) = 0$.

3. Suppose that $E \subseteq [0,1]$ is measurable and there exists a $\delta > 0$ such that

$$\big|E \cap [x-r, x+r]\big|_e \geq \delta\, r$$

for all $x \in (0,1)$ and $r > 0$ such that $(x-r, x+r) \subseteq [0,1]$. Prove that $|E| = 1$.

4. Assume that E is a measurable subset of $\mathbb{R}^d$ such that $|E| < \infty$.

(a) Suppose that $f\colon E \to [-\infty, \infty]$ is measurable and finite a.e. Given $\varepsilon > 0$, prove that there exists a closed set $F \subseteq E$ such that $|E \setminus F| < \varepsilon$ and f is bounded on F.

(b) For each $n \in \mathbb{N}$ let f_n be a measurable function on E, and suppose that

$$\forall\, x \in E, \quad M_x = \sup_{n \in \mathbb{N}} |f_n(x)| < \infty.$$

Prove that for each $\varepsilon > 0$, there exists a closed set $F \subseteq E$ and a finite constant M such that $|E \setminus F| < \varepsilon$ and $|f_n(x)| \leq M$ for all $x \in F$ and $n \in \mathbb{N}$.

5. Let $f : [0,1] \to \mathbb{R}$ be a monotone nondecreasing function. Assume f is differentiable almost everywhere.

(i) Prove that $\displaystyle\int_0^1 f'(x)dx \leq f(1) - f(0)$.

(ii) Let $(f_n)_n$ be a sequence of monotone nondecreasing functions on $[0,1]$ such that $F(x) = \sum_{n=1}^\infty f_n(x)$ converges for all $x \in [0,1]$. Show that $\sum_{n=1}^\infty f_n'(x)$ converges almost everywhere on $[0,1]$ and $F'(x) = \sum_{i=1}^\infty f_i'(x)$ a.e.

Hints: (i) Use Fatou's Lemma.

(ii) Set $R_n(x) = \sum_{k=n}^\infty f_n(x)$. Use that $R_n(1) - R_n(0) \to 0$ and (i) to show that $|R_n'(x)| \to 0$ almost everywhere.

6. Given $f \in L^1(\mathbb{R})$, define

$$g(x) = \int_{-\infty}^{x} f(t)\,dt, \qquad x \in \mathbb{R}.$$

Given $c > 0$, prove that $g(x+c) - g(x)$ is an integrable function of x, and show that

$$\int_{-\infty}^{\infty} \Big(g(x+c) - g(x)\Big)\,dx = c \int_{-\infty}^{\infty} f(t)\,dt.$$

7. Let $H = L^2(\mathbb{R})$ be the Hilbert space of square integrable functions on $\mathbb{R}$ and define $U : H \to H$ by

$$U(f)(x) = f(x-1)$$

for $f \in H$. Show that U has no nonzero eigenvectors.

8. Show that $f\colon [a,b] \to \mathbb{R}$ is Lipschitz if and only if f is absolutely continuous and $f' \in L^\infty[a,b]$.

Note: Give a direct proof that Lipschitz functions are absolutely continuous.