Analysis Comprehensive Exam
January 22, 2016

Student Number: [__]  

Instructions: Complete up to 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1 2 3 4 5 6 7 8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.

NOTES:

- All functions in this exam are (extended) real-valued.

- The exterior Lebesgue measure of $E \subseteq \mathbb{R}^d$ is denoted by $|E|_e$, and if $E$ is measurable then its Lebesgue measure is $|E|$.

- The characteristic function of a set $A$ is denoted by $\chi_A$.

- A consequence of the Stone–Weierstrass theorem is the Weierstrass Approximation Theorem, which states that the set of polynomials on $[0, 1]^d$ is dense in $C([0, 1]^d)$ if the latter set is endowed with the uniform norm. You can use this fact without proof.

- You can also use without proof the fact the every monotone function $f : [a, b] \to \mathbb{R}$ is Borel measurable and is differentiable almost everywhere.
1. Prove that \( E \subseteq \mathbb{R}^d \) is measurable if and only if \(|Q| = |Q \cap E| + |Q \setminus E|\), for every box \( Q \).

2. Let \( \mu \) be a positive, Borel regular measure on \( I = [0,1] \) such that \( \mu(I) = 1 \). Set \( \xi_n(x) = x^n \) for \( n = 0, 1, 2, \ldots \) and \( x \in I \). Let

\[
H = L^2(I, \mu), \quad V = L^\infty(I, \mu).
\]

Let \( \langle \cdot, \cdot \rangle_H \) denote the inner product on \( H \) and \( || \cdot ||_H \) the norm on \( H \). Let \( || \cdot ||_V \) denote the norm on \( V \).

(i) Show that \( ||\xi_n||_H \rightarrow 0 \) if and only if \( \mu\{1\} = 0 \).

(ii) Show that if \( f \in H \) and \( \langle f, \xi_n \rangle_H = 0 \) for all \( n = 0, 1, 2, \ldots \), then \( f = 0 \) \( \mu \)-almost everywhere.

(iii) Show that \( ||\xi_n||_V \rightarrow 0 \) if and only if for some \( \epsilon > 0 \) we have \( \mu([1 - \epsilon, 1]) = 0 \).

3. Suppose that \( E \subseteq [0, 1] \) is measurable and there exists a \( \delta > 0 \) such that

\[
|E \cap [x-r, x+r]|_e \geq \delta r
\]

for all \( x \in (0, 1) \) and \( r > 0 \) such that \( (x-r, x+r) \subseteq [0, 1] \). Prove that \( |E| = 1 \).

4. Assume that \( E \) is a measurable subset of \( \mathbb{R}^d \) such that \( |E| < \infty \).

   (a) Suppose that \( f : E \rightarrow [-\infty, \infty] \) is measurable and finite a.e. Given \( \epsilon > 0 \), prove that there exists a closed set \( F \subseteq E \) such that \( |E \setminus F| < \epsilon \) and \( f \) is bounded on \( F \).

   (b) For each \( n \in \mathbb{N} \) let \( f_n \) be a measurable function on \( E \), and suppose that

\[
\forall x \in E, \quad M_x = \sup_{n \in \mathbb{N}} |f_n(x)| < \infty.
\]

Prove that for each \( \epsilon > 0 \), there exists a closed set \( F \subseteq E \) and a finite constant \( M \) such that \( |E \setminus F| < \epsilon \) and \( |f_n(x)| \leq M \) for all \( x \in F \) and \( n \in \mathbb{N} \).

5. Let \( f : [0, 1] \rightarrow \mathbb{R} \) be a monotone nondecreasing function. Assume \( f \) is differentiable almost everywhere.

   (i) Prove that \( \int_0^1 f'(x) dx \leq f(1) - f(0) \).

   (ii) Let \( (f_n)_n \) be a sequence of monotone nondecreasing functions on \( [0, 1] \) such that \( F(x) = \sum_{n=1}^\infty f_n(x) \) converges for all \( x \in [0, 1] \). Show that \( \sum_{n=1}^\infty f'_n(x) \) converges almost everywhere on \( [0, 1] \) and \( F'(x) = \sum_{i=1}^\infty f'_i(x) \) a.e.

   **Hints:** (i) Use Fatou’s Lemma.

   (ii) Set \( R_n(x) = \sum_{k=n}^\infty f_n(x) \). Use that \( R_n(1) - R_n(0) \rightarrow 0 \) and (i) to show that \( |R'_n(x)| \rightarrow 0 \) almost everywhere.
6. Given $f \in L^1(\mathbb{R})$, define

$$g(x) = \int_{-\infty}^{x} f(t) \, dt, \quad x \in \mathbb{R}.$$ 

Given $c > 0$, prove that $g(x + c) - g(x)$ is an integrable function of $x$, and show that

$$\int_{-\infty}^{\infty} (g(x + c) - g(x)) \, dx = c \int_{-\infty}^{\infty} f(t) \, dt.$$ 

7. Let $H = L^2(\mathbb{R})$ be the Hilbert space of square integrable functions on $\mathbb{R}$ and define $U : H \to H$ by

$$U(f)(x) = f(x - 1)$$

for $f \in H$. Show that $U$ has no nonzero eigenvectors.

8. Show that $f : [a, b] \to \mathbb{R}$ is Lipschitz if and only if $f$ is absolutely continuous and $f' \in L^\infty[a, b]$.

Note: Give a direct proof that Lipschitz functions are absolutely continuous.