Analysis Comprehensive Exam
January 11, 2017

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

Notes.

i. Unless otherwise specified, functions are (extended) real-valued.

ii. The exterior Lebesgue measure of a set $E \subseteq \mathbf{R}^d$ is denoted by $|E|_e$. If E is Lebesgue measurable, then its Lebesgue measure is denoted by $|E|$.

iii. The characteristic function of a set A is denoted by $\mathbf{1}_A$.

1. Assume f is real-valued and has bounded variation on $[a, b]$, and extend f to $\mathbf{R}$ by setting $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$. Prove that there exists a constant $C > 0$ such that
$$\|T_t f - f\|_1 \le C|t|, \qquad t \in \mathbf{R},$$
where $T_t f(x) = f(x - t)$ denotes the translation of f by t.

2. Functions in this problem are complex-valued.

   Let $W(x) = \max\{1 - |x|, 0\}$ be the "hat function" on $[-1, 1]$. Given $f \in L^1(\mathbf{R})$, let
$$g(y) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i y t}\, dt.$$
   Prove that g is bounded on $\mathbf{R}$, and for a.e. x we have
$$\int_{-\infty}^{\infty} f(y) \left(\frac{\sin \pi(x-y)}{\pi(x-y)}\right)^2 dy = \int_{-1}^{1} g(t)\,(1 - |t|)\, e^{2\pi i t x}\, dt.$$
   Hint: $\int_{-\infty}^{\infty} W(t)\, e^{2\pi i y t}\, dt = \left(\frac{\sin \pi y}{\pi y}\right)^2$ by direct calculation (which you may assume without proof).

3. Let c_0 be the space of all real-valued sequences that vanish at infinity:
$$c_0 = \Big\{x = (x_k)_{k\in\mathbf{N}} : \lim_{k\to\infty} x_k = 0\Big\}.$$
   The norm on c_0 is the sup-norm,
$$\|x\|_\infty = \sup_{k\in\mathbf{N}} |x_k|.$$
   Prove directly that the dual space of c_0 is isometrically isomorphic to ℓ^1.

4. Let A be a measurable subset of $[0, 1]$.

   (a) Prove that if $|A| > \frac{2}{3}$, then A contains an arithmetic progression of length 3, that is, prove that there are $a, d \in \mathbf{R}$ such that $a, a + d, a + 2d \in A$.

   (b) Use part (a) to prove that if $|A| > 0$, then A contains an arithmetic progression of length 3.

5. Let $h(x) = \dfrac{1}{\sqrt{|\sin 2\pi x|}}$, and consider the function $H(x) = \displaystyle\sum_{k=1}^{\infty} \frac{h(kx)}{k^2}$.

   (a) Prove that $H = \infty$ on a dense subset of $\mathbf{R}$.

   (b) Prove that H converges to a finite number a.e. on $\mathbf{R}$.

6. Given $A \subseteq [\,0,1\,]$, prove that A is Lebesgue measurable if and only if

$$|A|_e + |[0,1] \setminus A|_e = 1.$$

7. Let $f_n : [0,1] \mapsto \mathbf{R}$ be nonnegative measurable functions that converge almost everywhere to a measurable function $f \in L^1[0,1]$.

   (a) Prove that the integrals $\displaystyle\int_0^1 \min\{f_n(x), f(x)\}\,dx$ are defined for each n, and

$$\lim_{n\to\infty} \int_0^1 \min\{f_n(x), f(x)\}\,dx = \int_0^1 f(x)\,dx.$$

   (b) Assume that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

   Use part (a) to prove that f_n converges to f in L^1-norm on $[0,1]$.

8. Let X be a set and let $\mathfrak{M}$ be a sigma algebra of subsets of X. Suppose that $(X, \mathfrak{M}, \mu)$ and $(X, \mathfrak{M}, \nu)$ are two finite measure spaces, with μ a positive measure and ν a signed measure. Prove that the following statements are equivalent.

   (a) ν is absolutely continuous with respect to μ ($\nu \ll \mu$), i.e. if $E \in \mathfrak{M}$ satisfies $\mu(E) = 0$ then $\nu(E) = 0$.

   (b) For every $\varepsilon > 0$ there exists some $\delta > 0$ such that if $E \in \mathfrak{M}$ satisfies $\mu(E) < \delta$ then $\nu(E) < \varepsilon$.