Analysis Comprehensive Exam
Spring 2018

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let $A \subseteq \mathbb{R}$ be a measurable set. For $x \in \mathbb{R}$ denote $A + x = \{a + x : a \in A\}$. Prove that if A satisfies
$$|A \setminus (A + x)| = 0 \qquad \forall x \in \mathbb{R},$$
then either $|A| = 0$ or $|\mathbb{R} \setminus A| = 0$. (Note that here $A \setminus B = \{x \in A : x \notin B\}$).

2. Let $E \subset \mathbb{R}^n$ be measurable with $|E| < \infty$, and let $f : E \to \mathbb{R}$, $f_k : E \to \mathbb{R}$ be measurable, $k \geq 1$. Assume that every subsequence of $\{f_k\}$ contains another subsequence that converges to f a.e. on E.
   (i) Prove that $\{f_k\}$ converges in measure to f on E.
   (ii) Prove the following extension of Lebesgue's Dominated Convergence Theorem: assume that there is an integrable function $\phi : E \to \mathbb{R}$ such that for $k \geq 1$,
   $$|f_k(x)| \leq \phi(x) \qquad \text{for a.e. } x \in I.$$
   Prove that f is integrable and
   $$\lim_{k\to\infty} \int_E f_k(x)\,dx = \int_E f(x)\,dx.$$

3. Let $g(x) = x^2 + 1 + \sin(2018x)$.
   i. Prove that the function $\phi : [0, \infty) \to [0, \infty)$ defined by $\phi(s) = |\{x : g(x) < s\}|$ is continuous.
   ii. Let
   $$\mathfrak{F} := \{f \in L^1(\mathbb{R}) : f : \mathbb{R} \to [0, 1] \text{ and } \int_{\mathbb{R}} f = 1\}.$$
   Prove that $\inf_{f\in\mathfrak{F}} \int_{\mathbb{R}} fg$ is obtained for a function f of the form $f = \mathbb{1}_{\{g<s\}}$ for some constant $s \in \mathbb{R}$.

4. Let $f : [0, 1] \to [0, 1]$ be defined by $f(0) = 0$ and
$$f(x) = x^2 \left|\sin \frac{1}{x}\right|, \qquad x \in (0, 1].$$
Show that f is absolutely continuous on $[0, 1]$. Give an example of a function $\phi : [0, 1] \to [0, 1]$ that is of bounded variation, and such that ϕ' exists in $(0, 1]$ but such that $\phi \circ f$ is not absolutely continuous in $[0, 1]$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function with continuous partial derivatives. Denote $U = [0, 1]^2 \subset \mathbb{R}^2$. Assume that $\partial f/\partial x$ and $\partial f/\partial y$ are Lipschitz functions which vanish on the boundary of U (that is, they are equal zero on the boundary).

i. Denote $h = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$. Prove that h is defined almost everywhere on U and that for every $(x, y) \in U$ we have

$$f(x,y) = f(x,0) + \int_{[0,x]\times[0,y]} h.$$

ii. Prove that for almost every $(x, y) \in U$ we have

$$\frac{\partial f}{\partial x}(x,y) = \int_{[0,y]} h(x,s)ds.$$

iii. Prove that the functions $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ and $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ are equal almost everywhere on U.

6. i. Let $E_k \subset \mathbb{R}^n$, $k \in \mathbb{N}$, be sets which satisfy $E_1 \subseteq E_2 \subseteq E_3 \subseteq ...$, and denote $E = \cup_k E_k$. Assume that $|E|_e$ is finite. Prove that

$$|E|_e = \lim_{k\to\infty} |E_k|_e.$$

Here the subscript e denotes the exterior Lebesgue measure.

ii. Let E be a set in $\mathbb{R}^n$ with $|E|_e$ finite and positive. Let $0 < \theta < 1$. Show that there is a set $E_\theta \subset E$ with

$$|E_\theta|_e = \theta\, |E|_e\,.$$

7. Suppose that μ, ν are probability measures on $[0, 1]$, and

$$\int_{[0,1]} t^j d\mu\,(t) = \int_{[0,1]} t^j d\nu\,(t)$$

for all $j \geq 0$. Assume also that $\mu(\{0\}) = \nu(\{0\})$. Prove that for every $d \in [0,1]$,

$$\mu([0,d]) = \nu([0,d])\,.$$

(Hint: you may assume Weierstrass' approximation theorem).

8. Let B be an **infinite dimensional** Banach space and let J be an index set. Assume that $\{x_j\}_{j\in J} \subseteq B$ is a Hamel basis for B, that is:

i. Every $y \in B$ can be written as a **finite** linear combination of vectors in $\{x_j\}$:

$$y = \sum_{j=1}^{N} \alpha_j x_j.$$

ii. The elements in B are linearly independent: If $\sum_{j=1}^{N} \alpha_j x_j = 0$ then $\alpha_j = 0$ for every j.

Prove that the set J is uncountable.