Differential equations Comprehensive Exam
Spring 2018

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

- In problems 1-3, $x' = \dot{x} = x_t = \frac{dx}{dt}$.

1. Consider the system of differential equations

$$\begin{cases} x' = x + 2y - 4x(x^2 + y^2) \\ y' = -2x + y - 4y(x^2 + y^2) \end{cases}$$

   (1) Let $x \in R^2$, what is the omega limit set $\omega(x)$ of x?

   (2) What is the omega limit set $\omega(B)$ of the unit disk B centered at the origin (with radius 1)?

   Prove your answers.

2. Consider the system of differential equations

$$x' = A(t)x, \quad x \in R^n$$

   where the $n \times n$ matrix $A(t)$ is C^0 and satisfies $A^T(t) = -A(t)$ for all t (A^T means transpose). Find all Lyapunov exponents of the system and prove your conclusion.

3. Consider the system of differential equations

$$\begin{cases} x' = -2xy \\ y' = -y + x^2 - y^2. \end{cases}$$

   (1) Find the local stable and center manifolds of the equilibrium $(0, 0)$. If you can not find the explicit formula of the functions, whose graphs give the local invariant manifolds, at least compute the coefficients in their Taylor expansions up to the cubic order.

   (2) Is the equilibrium $(0, 0)$ stable? Prove your conclusion.

4. Let $A_{n\times n}$ be invertible and none of its eigenvalues is on the unit circle S^1. Let C_b^0 denote the space of continuous and bounded functions from R^n to R^n. Prove that the operator T defined as

$$(Tf)(x) = Af(x) - f(Ax), \quad \forall x \in R^n$$

   is an isomorphism on C_b^0.

5. Denote by D the open unit disk in $\mathbf{R}^2$ centered at the origin and let $u \in C^2(D) \cap C(\overline{D})$ be a real function solving the boundary value problem

$$\begin{cases} \Delta u = u^3 \ \ \textbf{in} \ \ D \\ u = 0 \ \ \text{on} \ \ \partial D \end{cases}$$

   Show that u must vanish identically in D.

6. Given a smooth vector field $\vec{b}$ on $\mathbf{R}^n$ consider the equation

$$\begin{cases} u_t + \mathrm{div}(u\vec{b}) = 0 \quad \text{on} \quad \mathbf{R}^n \times (0,\infty) \\ u = g \quad \text{on} \quad \mathbf{R}^n \times \{t=0\} \ . \end{cases}$$

   Show that the solution is given by

$$u(x,t) = g(\phi_{-t}(x))\mathrm{det}D\phi_{-t}(x)$$

   where $\phi_t(x)$ is the one parameter flow associated with $\vec{b}$, i.e., $\phi_0(x) = x$ and $\frac{d}{dt}\phi_t(x) = \vec{b}(\phi_t(x))$. Hint: Show that

$$\frac{d}{dt}\mathrm{det}D\phi_t(x) = \mathrm{div}\vec{b}(\phi_t(x))\mathrm{det}D\phi_t(x)$$

   which also shows that the determinant can never vanish.

7. Solve

$$\begin{cases} u_{tt} - u_{xx} = x^2 \quad \text{on} \quad \mathbf{R} \times (0,\infty) \\ u = x \ , \ u_t = 0 \quad \text{on} \quad \mathbf{R} \times \{t=0\} \end{cases}$$

8. Show that there does not exist an analytic solution of the heat equation $u_t = u_{xx}$ in $\mathbf{R} \times \mathbf{R}$ with $u = \frac{1}{1+x^2}$ at $\{t=0\}$. Hint: Assume that such an analytic solution exists and compute the coefficients of its power series. Then show that this power series can only converge for $t = x = 0$.