Differential Equations Comprehensive Exam
January 18, 2017

Student Number: [ ]

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $u \in C^{2,1}(\Omega \times [0,T]) \cap C(\bar{\Omega} \times [0,T])$ be a solution to the equation

$$\begin{cases} u_t - \Delta u + x \cdot \nabla_x u = 0 & \text{in } \Omega \times (0,T), \\ u(x,t) = g(x,t) & \text{on } \partial\Omega \times [0,T] \ \cup \ \Omega \times \{t=0\}, \end{cases}$$

where $0 \le g \le 1$. Show that $u \le 1$ in $\Omega \times [0,T]$.

2. Suppose that $f \in C^1(\mathbb{R})$ is a bounded and strictly increasing function such that

$$\sup_{x \in \mathbb{R}} f'(x) = K > 1 \quad \text{with } K < \infty.$$

Consider the initial value problem

$$\begin{cases} u_t - uu_x + u = 0 & \text{in } \mathbb{R} \times (0,T), \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Find the smallest T (in terms of K) such that no classical C^1 solution exists for $t > T$, and write down the solution $u(x,t)$ (in implicit form) for $t \in [0,T)$.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Suppose $g \in C^\infty(\bar{\Omega})$ satisfies $g \ge 1$ in $\bar{\Omega}$. Given $f_1, f_2 \in C_c^\infty(\Omega)$, show that there can be at most one solution $u \in C^2(\bar{\Omega} \times [0,\infty))$ to the equation

$$\begin{cases} u_{tt} - g(x)\Delta u = 0 & \text{for } x \in \Omega, t > 0, \\ u(x,t) = 0 & \text{for } x \in \partial\Omega, t > 0, \\ u(x,0) = f_1(x), \ u_t(x,0) = f_2(x) & \text{for } x \in \partial\Omega. \end{cases}$$

4. Let Ω be the punctured closed unit disk in $\mathbb{R}^2$, i.e. $\Omega = \{x \in \mathbb{R}^2 : 0 < |x| \le 1\}$. Suppose that $u \in C^2(\Omega)$ is harmonic in the interior of Ω. Prove that if u is bounded in Ω, then $\sup_\Omega u \le \max_{\{|y|=1\}} u(y)$. Also give a counterexample to show that the conclusion can be false without the assumption that u is bounded.

5. Consider the differential equation

$$\frac{d}{dt} y(t) = f(t,y).$$

Assume that $f \in C(\mathbb{R} \times \mathbb{R})$, and there exists some constant $K > 0$, such that

$$f(t, y+h) - f(t,y) \le Kh \quad \text{for all } t \in \mathbb{R}, y \in \mathbb{R}, h > 0.$$

(Note that f is NOT necessarily Lipschitz in y as the above inequality requires $h > 0$.) If y_1, y_2 are two C^1 solutions to the equation in $[0,T]$ with $y_1(0) = y_2(0)$, prove that $y_1 \equiv y_2$ on $[0,T]$.

6. Consider the differential equation

$$\dot{x} = f(x)$$

where $x \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ is C^1. Let γ be a periodic orbit. If $\omega(x)$ is the ω-limit set of x, show that $\Omega(\gamma) = \{x \notin \gamma \,|\omega(x) = \gamma\}$ is open. Is it true that $\widehat{\Omega}(\gamma) = \{x \,|\omega(x) = \gamma\}$ is open?

7. Consider the equation

$$\dot{x} = f(x) + \epsilon h(t, x) \tag{1}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^{n+1} \to \mathbb{R}^n$ are C^2 functions, and $h(t, x)$ is T-periodic in t. Finally ϵ is a parameter. Assume that there exists x^* such that $f(x^*) = 0$ and

$$A = \frac{\partial f(x)}{\partial x}\Big|_{x=x^*}$$

is an invertible matrix. Prove that, for ϵ small enough, there exists a periodic solution $x(t, \epsilon)$ of (1) with $x(t, 0) \equiv x^*$ and

$$x(t, \epsilon) - x^* = O(\epsilon).$$

Compute

$$\delta x(t) = \frac{\partial x(t, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}.$$

8. Consider the linear differential equation

$$\begin{cases} \dot{x}_1 = \alpha(t)x_1 + \beta(t)x_2 \\ \dot{x}_2 = \gamma(t)x_1 + \delta(t)x_2 \end{cases}$$

where α, β, γ, and δ are continuous functions and

$$\liminf_{t\to\infty} \int_0^t (\alpha(s) + \delta(s))ds > -\infty$$

Suppose that there exists a solution $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t))$ such that $\bar{x}(0) \neq (0, 0)$ and $\lim_{t\to\infty} |\bar{x}(t)| = 0$. Show that for any solution $x(t)$ such that $x(0) \neq \lambda\bar{x}(0)$, for any $\lambda \in \mathbb{R}$, we have $\lim_{t\to\infty} |x(t)| = \infty$.