Discrete Mathematics Comprehensive Exam
January 22, 2016

Student Number: ☐

Instructions: Complete up to 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let G be a nonempty graph (without loops) and assume that any two odd cycles in G intersect. Show that $\chi(G) \leq 5$ and give an example to show that is bound is tight.

2. Prove that every 2-edge-connected cubic graph contains a perfect matching.

3. Show that if G is a 2-edge-connected plane graph and contains a Hamilton cycle then G is 4-face-colorable.

4. Let G be a connected graph. Use a depth-first-search spanning tree to prove that if G is triangle-free then G contains a bipartite subgraph H such that $|E(H)| \geq 3(|V(G)|-1)/4$ and every component of H is an induced subgraph of G.

5. Let n be a positive integer, and let $\mathbf{2}^n$ denote the subset lattice consisting of subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion. A family $\mathcal{F}$ of subsets of $\{1, 2, \ldots, n\}$ is called a *down set* in $\mathbf{2}^n$ if $S \in \mathcal{F}$ whenever $T \in \mathcal{F}$ and $S \subseteq T$. If $\mathcal{F}$ and $\mathcal{G}$ are down sets in $\mathbf{2}^n$, show that $|\mathcal{F}||\mathcal{G}| \leq |\mathcal{F} \cap \mathcal{G}|2^n$.

6. Let G be a graph on n vertices and let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of G. Use a probabilistic argument to show that G has an independent set whose size is at least $\sum_{i=1}^{n} 1/(d_i + 1)$.

7. Let $\mathcal{H}$ be a 3-uniform simple hypergraph for which each vertex x belongs to at most k hyper-edges. Trivially, the chromatic number of $\mathcal{H}$ is at most $k+1$. Use the Lovász Local Lemma to show that the chromatic number of $\mathcal{H}$ is $O(\sqrt{k})$.

8. Let n be a positive integer. Then let a_n be the number of partitions of the integer n so that the number of parts of size j is less than j for each $j \geq 1$. Also, let b_n be the number of partitions of n so that no part size is a perfect square. An example of a partition of the first type is $17 = 4 + 4 + 4 + 3 + 2$ and an example of a partition of the second type is $39 = 10 + 8 + 8 + 5 + 3 + 3 + 2$. Show that $a_n = b_n$ for all $n \geq 1$.