Discrete Mathematics Comprehensive Exam
Spring 2018

Student Number: ☐

Instructions: Complete 5 of the 7 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Prove that if G is a graph with $m \geq 1$ edges then G has a bipartite subgraph with more than $m/2$ edges.

2. Show that if $\chi(G) \geq 4$ then G contains a subdivision of K_4.

3. Let G be a 3-connected graph. Show that G contains an induced non-separating cycle. (A cycle in a connected graph is non-separating if its deletion results in a connected graph.)

4. Let $r \geq 1$ be an integer and G an r-regular graph. Show that G is bipartite if and only if $E(G)$ can be decomposed into (edge-disjoint) copies of $K_{1,r}$.

5. Let M and N be finite sets of size $|M| = m$ and $|N| = n$. Trivially, there exists an injection from M to N if $n \geq m$. The goal of this exercise is to prove a slightly weaker statement, demonstrating the strength of the Lovász Local Lemma (LLL) compared to the union bound/first moment method.

   (a) Using a union bound argument/first moment method argument, show that an injective mapping $f : M \to N$ exists if $n > \binom{m}{2}$.

   (b) Using the LLL, show that an injective mapping $f : M \to N$ exists if $n > 6m$.

6. By $G(n,p)$ we denote the binomial random graph with vertex set $[n]$, where each of the $\binom{n}{2}$ edges of the complete graph K_n is included, independently, with probability p.

   (a) Show that that if $p \gg (\log n)^{1/2} n^{-1/2}$, then whp every edge of $G(n,p)$ is contained in at least one triangle.

   (b) Show that if $p \gg (\log n)^{1/3} n^{-2/3}$, then whp every vertex of $G(n,p)$ is contained in at least one triangle.

7. By $G(n,p)$ we denote the binomial random graph with vertex set $[n]$. Let $\omega = \omega(n) \to \infty$ as $n \to \infty$. For any $p = p(n) \in [0,1]$ show that

$$\mathbb{P}(G(n,p) \text{ is connected}) \to \begin{cases} o(1) & \text{if } p \leq (\log n - \omega)/n, \\ 1 - o(1) & \text{if } p \geq (\log n + \omega)/n. \end{cases}$$

   (It's not needed, but it's OK to assume that $\omega = \omega(n) \to \infty$ as $n \to \infty$ arbitrarily slowly, say, $\omega = o(\log \log \log n)$ holds, if this helps your calculations.)