Numerical Analysis Comprehensive Exam
Spring 2018

Student Number: ☐

Instructions: Complete 5 of the 7 problems, and circle their numbers below – the uncircled problems will not be graded.

1 2 3 4 5 6 7

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.
1. Find a Gauss quadrature formula that approximates \( \int_{0}^{h} f(x) \, dx \approx c_1 f(0) + c_2 f(c_3) \), where \( c_1, c_2 \) and \( c_3 \in (0, h] \) are to be determined. Estimate the order of accuracy (in terms of \( O(h^r) \)) of this approximation.

2. Let \( \{A_1, A_2, \ldots, A_6\} \) be the vertices and edge midpoints of a triangle in counterclockwise direction, beginning with a vertex. Let \( P^2 \) be the linear space of polynomials of degree up to two. One can define a set of Lagrangian basis functions of \( P^2 \) associated with these points. Find the Lagrangian basis functions of \( P^2 \) associated with \( A_1 \) and \( A_2 \).

3. Consider an initial value problem \( x' = f(x) \) with initial condition \( x(0) = x_0 \), where \( f \) is smooth enough function. What is the order of accuracy for the following scheme:

\[
x_{k+1} = x_k + \frac{h}{2}(3f_k - f_{k-1})
\]

Find the interval for \( h \) that the scheme is absolutely stable. If the scheme is not stable, explain why.

4. Consider the scheme

\[
\frac{U_{i+1}^n - U_i^n}{\Delta t} = \theta \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{\Delta x^2} + (1 - \theta) \frac{U_i^n - 2U_i^{n+1} + U_{i-1}^n}{\Delta x^2},
\]

\( 0 \leq \theta \leq 1 \). Analyze its stability for various \( \theta \).

5. Consider the Poisson equation \( -\Delta u = f \) defined on a polygonal domain \( \Omega \) with Dirichlet boundary condition \( u|_{\partial \Omega} = g \). Let \( \Omega \) be partitioned with a triangular mesh (without "hanging nodes"). Describe a piecewise linear continuous finite element method for solving this problem on the mesh and prove the existence and uniqueness of the numerical solution of the method.

6. Consider a symmetric matrix \( A \) given by

\[
A = \begin{bmatrix}
1 & 0.1 & 0 & 0 & 0 & \cdots & 0 \\
0.1 & 2 & 0.1 & 0 & 0 & \cdots & 0 \\
0 & 0.1 & 3 & 0.1 & 0 & \cdots & 0 \\
0 & 0 & 0.1 & 4 & 0.1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0.1 & 100
\end{bmatrix}
\]

Let \( \vec{x}_0 = \frac{1}{10}[1, \ldots, 1]^T \in \mathbb{R}^{100} \) and \( \mu = 50.25 \). One computes \( \vec{x}_n \) and \( \lambda_n \) by the following procedure,

\[
\text{for } n = 1, 2, \ldots \\
\quad \vec{w} = (A - \mu I)^{-1}\vec{x}_{n-1}; \\
\quad \vec{x}_n = \vec{w}/\|\vec{w}\|_2; \\
\quad \lambda_n = \vec{x}_n^T A \vec{x}_n; \\
\text{end;}
\]
Are $\vec{x}_n$ and $\lambda_n$ convergent? If so, what are they converge to and what are the rate of convergence? If not, explain why.

7. Consider to solve a linear system of equation $A\vec{x} = \vec{b}$ by iterative methods, where $A$ is a $m \times m$ real symmetric positive definite matrix, denote its solution as $\vec{x}^*$. Design a numerical algorithm that finds $\vec{x}_n \in \vec{x}_{n-1} + \text{span}\{\vec{r}\}$, where $\vec{r} = \vec{b} - A\vec{x}_{n-1}$, such that the new residual at $\vec{x}_n$ is orthogonal to $\vec{r}$. Prove that $\vec{x}_n$ is the minimizer of

$$
\min_{\vec{x} \in \vec{x}_{n-1} + \text{span}\{\vec{r}\}} \frac{[(\vec{x}^* - \vec{x})^T A(\vec{x}^* - \vec{x})]^{\frac{1}{2}}}{2}.
$$