Numerical Analysis Comprehensive Exam
January 18, 2017

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Consider the equation $u_t + a(x,t)u_x = 0$, where $a(x,t)$ is sufficiently smooth. Solving it on a uniform mesh with the CIR scheme, show that it has second order local error (with $\Delta t = C\Delta x$, $C > 0$ is a constant) and is unconditionally stable. The CIR scheme is defined as

$$U_i^{n+1} = U^n(x_i - a(x_i, t_{n+1})\Delta t)$$

in which the right-hand-side is a linear interpolation at the indicated location using neighboring grid point values at the time level t_n.

2. Conside the equation $u_t + au_x + bu_y = 0$, where a and b are constants. Solve the equation on a uniform rectangular mesh with the scheme

$$\frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t} + \frac{a}{2}\{\frac{U_{i+1,j}^{n+1} - U_{i-1,j}^{n+1}}{2\Delta x} + \frac{U_{i+1,j}^n - U_{i-1,j}^n}{2\Delta x}\} +$$

$$\frac{b}{2}\{\frac{U_{i,j+1}^{n+1} - U_{i,j-1}^{n+1}}{2\Delta y} + \frac{U_{i,j+1}^n - U_{i,j-1}^n}{2\Delta y}\} = 0.$$

What's the order of the scheme ? Analyze its stability. Tranform it into an ADI scheme.

3. Use the cubic Hermite interpolation polynomial for a sufficiently smooth function $f(x)$ to obtain

$$\int_a^b f(x)dx \approx \frac{b-a}{2}[f(a) + f(b)] - \frac{(b-a)^2}{12}[f'(b) - f'(a)].$$

Derive an error formula.

4. Let $f(x)$ be a sufficiently smooth function defined on $[a,b]$. Show that the Simpson's rule is 5th order accurate, i.e.

$$\int_a^b f(x)dx = \frac{h}{6}\{f(a) + 4f(\frac{a+b}{2}) + f(b)\} + O(h^5),$$

where $h = b - a$.

5. Let $f(x) = (x-\alpha)^p h(x)$, where h is a sufficiently smooth function, $h(\alpha) \neq 0$, p is a positive integer, and $p \geq 2$. Analyze the rate of convergence of the Newton's method for finding the solution α of $f(x) = 0$.

6. Consider the equation
$$\frac{dy}{dt} = f(t, y), \quad y(0) = Y_0.$$
Suppose f is sufficiently smooth and satisfies the Lipschitz condition $|f(t, y_1) - f(t, y_2)| \le k|y_1 - y_2|$ for some constant k, and the solution $y(t)$ has bounded second derivatives. Partition $[0, T]$ into N uniform subintervals and derive an error formula of $|y_N - y(t_N)|$, where y_N is the numerical solution by the Euler's method at the time level $t = t_N = T$.

7. Suppose A and B are n by n real matrices. Write an algorithm for computing n by n orthogonal matrices Q and Z such that $Q^T AZ$ is upper Hessenberg and $Z^T BQ$ is upper triangular.

8. Let $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{bmatrix}$ with $\epsilon = 10^{-9}$.

   (a) Solve the least square problem $Ax = b$ by method of your choice.

   (b) What happens if you solve the same problem by computer with standard double precision using the exact same method that you just did in (a). Explain what you may observe.

   (c) If you will solve this problem by Matlab (or any other computer language at your choice), describe an algorithm that will solve this least square problem.