Probability Comprehensive Exam
January 15, 2016

Student Number: ☐

Instructions: Complete up to 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let X have $\mathbb{E}X = 0$ and Var $X = \sigma^2 > 0$. Show that if $c > 0$, then

$$\mathbb{P}(X > c) \leq \frac{\sigma^2}{\sigma^2 + c^2}.$$

Hint: write $c - X = (c - X)_+ - (c - X)_-$ and use Cauchy-Schwarz type arguments.

2. Let $X = (X_1, X_2)$ be a Gaussian vector with zero mean and covariance matrix

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},$$

where $|\rho| < 1$. Find a matrix A such that $X = AZ$, where Z is a standard normal vector and derive the characteristic function of X as a function of ρ.

3. Let $X_1, X_2, \ldots$ be i.i.d. uniform $(0, 1)$ random variables. Show that

$$(X_1 \cdot \cdots \cdot X_n)^{1/n}$$

converges almost surely as $n \to \infty$ and compute the limit.

4. Let $X_1, X_2, \ldots$ be i.i.d. exponential variables with parameter 1 and set

$$M_n = \max\{X_1, \ldots, X_n\}.$$

Find sequences (a_n) and (b_n) of real numbers such that $(M_n - a_n)/b_n$ converges in distribution.

5. Let $(N_t)_{t \geq 0}$ be a rate-λ Poisson process. Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbb{E}|X_1| < \infty$ (independent of the Poisson process as well) and define

$$S_t = \sum_{i=1}^{N_t} X_i.$$

Show that S_t/t converges in probability to a constant and compute this constant.

6. Let $X_1, X_2, \ldots$ be i.i.d. standard normal random variables and for $x \in (-1, 1)$, set

$$Y = \sum_{n=1}^{\infty} x^n X_n.$$

Show that the sum defining Y converges and find its distribution.

7. Let $X_1, X_2, \ldots$ be independent random variables such that X_n has Binomial(n, p_n) distribution, for some $p_n > 0$. Show that if $np_n(1-p_n) \to \infty$, then

$$\frac{X_n - np_n}{\sqrt{np_n(1-p_n)}} \Rightarrow N(0,1).$$

8. A sequence of events $A_1, A_2, \ldots$ is said to be 1-dependent if for every $k \geq 1$, the sigma-algebras $\sigma(A_1, \ldots, A_k)$ and $\sigma(A_{k+2}, A_{k+3}, \ldots)$ are independent. Prove that if $A_1, A_2, \ldots$ are 1-dependent and E is a tail event:

$$E \in \cap_n \sigma(A_n, A_{n+1}, \ldots),$$

then $\mathbb{P}(E) = 0$ or 1.