Probability Comprehensive Exam
January 18, 2017

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Show that if X_n and Y_n are independent for $n = 1, 2, ...$ and $X_n \to^w X$, $Y_n \to^w Y$, where X and Y are independent, then $X_n + Y_n \to^w X + Y$.

2. Let X be a random variable with mean zero and finite variance σ^2. Prove that for every $c > 0$,
$$P(X > c) \leq \frac{\sigma^2}{\sigma^2 + c^2}.$$
Hint: Combine the inequality $\mathbb{E}(c - X) \leq \mathbb{E}\left((c - X)\mathbf{1}_{\{X<c\}}\right)$ with the Cauchy-Schwartz inequality.

3. Let $X_1, X_2, ...$ be i.i.d. random variables uniformly distributed on $[0, 1]$. Show that with probability 1,
$$\lim_{n\to\infty} (X_1 \cdot \cdots \cdot X_n)^{\frac{1}{n}}$$
exists and compute its value.

4. Let X and Y be independent and suppose that each has a uniform distribution on $(0, 1)$. Let $Z = \min\{X, Y\}$. Find the density $f_Z(z)$ for Z.

5. Show that the characteristic function φ of a random variable X is real if and only if X and $-X$ have the same distribution.

6. Let X_i be i.i.d. random variables uniformly distributed on $[0, 2]$. Let $S_n = X_1 + \cdots + X_n$. Show that
$$\frac{3\sqrt{3}}{2} n^{\frac{1}{6}} \left(\sqrt[3]{S_n} - \sqrt[3]{n}\right) \to^w Z,$$
where Z is a standard normal random variable.

7. Let $v = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$ be a unit vector in $\mathbf{R}^n$. Consider the set A in $\mathbf{R}^n$ be given by
$$A = \left\{x \in \mathbf{R}^n : \, x_i \in \left[-\frac{1}{2}, \frac{1}{2}\right], \langle x, v \rangle \leq \frac{t}{2\sqrt{3}}\right\}.$$
Prove that as the dimension $n \to \infty$,
$$Vol_n(A) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{x^2}{2}} dx + O(\frac{1}{\sqrt{n}}).$$

8. Assume $X_1, X_2, ..., X_n, ...$ are i.i.d. standard normal random variables. Show without using the law of the iterated logarithm that for any $\lambda > 1/2$,
$$\frac{1}{n^\lambda}(X_1 + ... + X_n) \to^{a.s.} 0$$