Probability Comprehensive Exam  
Spring 2018

Student Number: 

Instructions: Complete 5 of the 9 problems, and circle their numbers below – the uncircled problems will not be graded.

1  2  3  4  5  6  7  8  9

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.
1. Let \( \{X_n\} \) be a sequence of independent identically distributed random variables with exponential distribution (in other words, \( X_n \geq 0 \text{ a.s. and } P\{X_n \geq t\} = e^{-\lambda t}, t \geq 0 \) for some \( \lambda > 0 \)). Prove that
\[
\limsup_{n \to \infty} \frac{X_n}{\log n} < \infty \text{ a.s.}
\]

2. Suppose \( f \) is a continuous function on \([0, 1]\). Use the Law of Large Numbers to prove that
\[
\lim_{n \to \infty} \int_0^1 \cdots \int_0^1 f((x_1 \ldots x_n)^{1/n})dx_1 \ldots dx_n = f\left(\frac{1}{e}\right).
\]

3. Let \( X, Y \) be random variables with \( \mathbb{E}|X| < \infty, \mathbb{E}|Y| < \infty \). If \( \mathbb{E}(X|Y) = Y \) and \( \mathbb{E}(Y|X) = X \) a.s., then \( X = Y \) a.s. Prove it.

4. Let \( X_1, \ldots, X_n \) be i.i.d. random variables with mean \( \mu \) and variance \( \sigma^2 < +\infty \). Let \( f \) be a function continuously differentiable at the point \( \mu \). Prove that the sequence of random variables
\[
n^{1/2}\left(f\left(\frac{X_1 + \cdots + X_n}{n}\right) - f(\mu)\right)
\]
converges in distribution to a normal random variable. What is the mean and the variance of the limit?

5. Let \( X_1, \ldots, X_n, \ldots \) be i.i.d. random variables with \( \mathbb{E}X_1 = 0 \) and \( \text{Var}(X_1) = 1 \). Let \( S_n = X_1 + \cdots + X_n \). Prove that
\[
\limsup_{n \to \infty} \frac{S_n}{\sqrt{n}} = +\infty.
\]

6. Let \( (X_n) \) be an i.i.d. sequence of random variables with
\[
\mathbb{P}(X_n = 1) = 1/2 = \mathbb{P}(X_n = -1).
\]
Let \( (Y_n) \) be a bounded sequence of random variables such that \( \mathbb{P}(Y_n \neq X_n) \leq e^{-n} \). Show that
\[
\frac{1}{n} \mathbb{E}(Y_1 + \cdots + Y_n)^2 \to 1 \text{ as } n \to \infty.
\]

7. Let \( F_n, F \) be distribution functions such that \( F_n \to F \) weakly. If \( F \) is continuous, show that
\[
\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \to 0.
\]
8. Let \((X_n)\) be an i.i.d. sequence of random variables. Show that \(E(X_1)^2 < \infty\) if and only if for every \(c > 0\), \(\mathbb{P}(|X_n| \geq c\sqrt{n} \text{ infinitely often}) = 0\).

9. Find an example of a random variable \(X\) with a density function but whose characteristic function \(\phi_X\) satisfies

\[
\int_{-\infty}^{\infty} |\phi_X(t)| \, dt = \infty.
\]