Probability Comprehensive Exam
Spring 2018

Student Number: ☐

Instructions: Complete 5 of the 9 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8 9

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let $\{X_n\}$ be a sequence of independent identically distributed random variables with exponential distribution (in other words, $X_n \geq 0$ a.s. and $\mathbb{P}\{X_n \geq t\} = e^{-\lambda t}, t \geq 0$ for some $\lambda > 0$). Prove that
$$\limsup_{n\to\infty} \frac{X_n}{\log n} < \infty \text{ a.s.}$$

2. Suppose f is a continuous function on $[0,1]$. Use the Law of Large Numbers to prove that
$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 f((x_1 \ldots x_n)^{1/n}) dx_1 \ldots dx_n = f\Big(\frac{1}{e}\Big).$$

3. Let X, Y be random variables with $\mathbb{E}|X| < \infty, \mathbb{E}|Y| < \infty$. If $\mathbb{E}(X|Y) = Y$ and $\mathbb{E}(Y|X) = X$ a.s., then $X = Y$ a.s. Prove it.

4. Let $X_1, \ldots, X_n$ be i.i.d. random variables with mean μ and variance $\sigma^2 < +\infty$. Let f be a function continuously differentiable at the point μ. Prove that the sequence of random variables
$$n^{1/2}\left(f\left(\frac{X_1 + \cdots + X_n}{n}\right) - f(\mu)\right)$$
converges in distribution to a normal random variable. What is the mean and the variance of the limit?

5. Let $X_1, \ldots, X_n, \ldots$ be i.i.d. random variables with $\mathbb{E}X_1 = 0$ and $\mathrm{Var}(X_1) = 1$. Let $S_n = X_1 + \cdots + X_n$. Prove that
$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = +\infty.$$

6. Let (X_n) be an i.i.d. sequence of random variables with
$$\mathbb{P}(X_n = 1) = 1/2 = \mathbb{P}(X_n = -1).$$
Let (Y_n) be a bounded sequence of random variables such that $\mathbb{P}(Y_n \neq X_n) \leq e^{-n}$. Show that
$$\frac{1}{n}\mathbb{E}(Y_1 + \cdots + Y_n)^2 \to 1 \text{ as } n \to \infty.$$

7. Let F_n, F be distribution functions such that $F_n \to F$ weakly. If F is continuous, show that
$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \to 0.$$

8. Let (X_n) be an i.i.d. sequence of random variables. Show that $\mathbb{E}(X_1)^2 < \infty$ if and only if for every $c > 0$, $\mathbb{P}(|X_n| \geq c\sqrt{n} \text{ infinitely often}) = 0$.

9. Find an example of a random variable X with a density function but whose characteristic function ϕ_X satisfies

$$\int_{-\infty}^{\infty} |\phi_X(t)| \, \mathrm{d}t = \infty.$$