Topology Comprehensive Exam
August 31, 2016

Student Number: [ ]

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1  2  3  4  5  6  7  8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.
1. Let $n < m$ be positive integers. Use Sard’s theorem to show that any continuous map from $S^n$ to $S^m$ is homotopic to a constant map.

2. Let $S$ be a smooth submanifold of a smooth manifold $M$, and let $X, Y$ be vector fields on $M$ that are tangent to $S$. Show that $[X, Y]$ is also tangent to $S$.

3. Recall that the standard embedding of $RP^1$ in $RP^2$ is the image of the equator under the 2-fold cover $\pi : S^2 \to RP^2$ given by $\pi(x) = \pi(-x)$. Let $X$ be the union of two real projective planes glued via the identity map of the standardly embedded $RP^1$s. Compute the fundamental group of $X$.

4. Let $p \in \mathbb{R}^3$ be a point outside the $x$-axis. Let $X$ be the complement of the $x$-axis in $\mathbb{R}^3 \setminus \{p\}$. For a positive integer $k$ let $X_k$ be a $k$-sheeted covering space of $X$. Show that $X_k$ is homeomorphic to $S^1 \times \mathbb{R}^2$ with $k$ points removed.

5. Let $f : M \to N$ be a smooth map without critical points, where $M, N$ are connected $n$-dimensional smooth manifolds without boundary, and $M$ is compact. Show that the induced map $f_* : \pi_1(M) \to \pi_1(N)$ is injective. Does the statement hold if $M$ is not compact?

6. Let $D^2 = \{x \in \mathbb{R}^2 : |x| \leq 1\}$ and $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$. Let $V$ be a smooth vector field on $X = D^2 \times S^1$ such that

   (i) if $x \in D^2 \times \{t\}$, then $V(x)$ is not tangent to $D^2 \times \{t\}$,
   
   (ii) if $x \in \partial X$, then $V(x)$ is tangent to $\partial X$.

Show that $V$ has a closed orbit, i.e. there is a flow line $f : \mathbb{R} \to X$ of $V$ such that $f(t + P) = f(t)$ for some $P \in \mathbb{R}$ and all $t \in \mathbb{R}$.

7. Suppose $\alpha$ is a closed 2-form on a 4-dimensional sphere $S^4$. Show that the 4-form $\alpha \wedge \alpha$ vanishes at some point $x \in S^4$.

8. Let $Y$ be the wedge of two circles $a$ and $b$, and let $X$ be connected covering space of $Y$. Assume that among the lifts of $a$ exactly one is a loop. Show that the every deck transformation of the covering $X \to Y$ is trivial.