Topology Comprehensive Exam
August 31, 2016

Student Number: [ ]

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let $n < m$ be positive integers. Use Sard's theorem to show that any continuous map from S^n to S^m is homotopic to a constant map.

2. Let S be a smooth submanifold of a smooth manifold M, and let X, Y be vector fields on M that are tangent to S. Show that $[X, Y]$ is also tangent to S.

3. Recall that the standard embedding of RP^1 in RP^2 is the image of the equator under the 2-fold cover $\pi : S^2 \to RP^2$ given by $\pi(x) = \pi(-x)$. Let X be the union of two real projective planes glued via the identity map of the standardly embeded RP^1s. Compute the fundamental group of X.

4. Let $p \in \mathbf{R}^3$ be a point outside the x-axis. Let X be the complement of the x-axis in $\mathbb{R}^3 \setminus \{p\}$. For a positive integer k let X_k be a k-sheeted covering space of X. Show that X_k is homeomorphic to $S^1 \times \mathbf{R}^2$ with k points removed.

5. Let $f : M \to N$ be a smooth map without critical points, where M, N are connected n-dimensional smooth manifolds without boundary, and M is compact. Show that the induced map $f_* : \pi_1(M) \to \pi_1(N)$ is injective. Does the statement hold if M is not compact?

6. Let $D^2 = \{x \in \mathbf{R}^2 : |x| \leq 1\}$ and $S^1 = \{x \in \mathbf{R}^2 : |x| = 1\}$. Let V be a smooth vector field on $X = D^2 \times S^1$ such that

   (i) if $x \in D^2 \times \{t\}$, then $V(x)$ is not tangent to $D^2 \times \{t\}$,

   (ii) if $x \in \partial X$, then $V(x)$ is tangent to ∂X.

   Show that V has a closed orbit, i.e. there is a flow line $f : \mathbf{R} \to X$ of V such that $f(t + P) = f(t)$ for some $P \in \mathbf{R}$ and all $t \in \mathbf{R}$.

7. Suppose α is a closed 2-form on a 4-dimensional sphere S^4. Show that the 4-form $\alpha \wedge \alpha$ vanishes at some point $x \in S^4$.

8. Let Y be the wedge of two circles a and b, and let X be connected covering space of Y. Assume that among the lifts of a exactly one is a loop. Show that the every deck transformation of the covering $X \to Y$ is trivial.