Topology Comprehensive Exam
Spring 2018

Student Number:

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. We can regard $\pi_1(X, x_0)$ as base point preserving homotopy classes of maps of (S^1, pt) to (X, x_0). Let $[S^1, X]$ be the set of homotopy classes of maps S^1 to X (not necessarily base point preserving). There is a natural map

$$\Psi : \pi_1(X, x_0) \to [S^1, X]$$

that just ignores the base points. Show that Ψ is onto if X is path connected. Also show that $\Psi([\gamma]) = \Psi([\lambda])$ if and only if there is some $g \in \pi_1(X, x_0)$ such that $[\gamma] = g^{-1}[\lambda]g$.

2. Let G and H be groups and X and Y topological spaces such that $\pi_1(X, x_0) \cong G$ and $\pi_1(Y, y_0) \cong H$. If X is a finite connected 2-complex, then show that for any homomorphism $\phi : G \to H$ there is continuous map $f : X \to Y$ such that $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is given by ϕ under the isomorphisms above.

3. Use algebraic topology to show that the free group of rank 2 has a normal subgroup of index 3 and a non-normal subgroup of index 3.

4. Given two points x and y in a connected smooth manifold M, show there is a compactly supported isotopy of M from $\phi_0 : M \to M$ to $\phi_1 : M \to M$ such that ϕ_0 is the identity on M and $\phi_1(x) = y$.
Hint: If helpful, you may assume without proof that in a connected manifold any two points can be connected by a smooth injective path with nonzero derivative at each point.

5. Let $f : M \to N$ and $g : N \to M$ be two smooth maps between compact oriented manifolds (without boundary) of the same dimension. If N is connected and $g \circ f$ is a diffeomorphism show that f and g are both diffeomorphisms.

6. Let $M(n, \mathbf{R})$ be the set of all n by n matrices (recall that it can be identified with $\mathbf{R}^{n^2}$ by choosing an ordering of the entries of the matrix). Let $O(n)$ be the subset of $M(n, \mathbf{R})$ consisting of matrices satisfying $A^tA = Id$ where A^t is the transpose of A and Id is the identity matrix. Show that $O(n)$ is a manifold and compute its dimension. Describe the tangent space to the identity of $O(n)$.

7. Let S_1 and S_2 be two submanifolds of M (all manifolds are without boundary). Define what it means for S_1 to be transverse to S_2 and if they are transverse show carefully that $S_1 \cap S_2$ is a submanifold of M of dimension $dim(S_1) + dim(S_2) - dim(M)$.

8. Let $\alpha_1, \ldots, \alpha_k$ be 1-forms on a smooth manifold M show that they are linearly independent if and only if at some point $\alpha_1 \wedge \ldots \wedge \alpha_k \neq 0$.