Topology Comprehensive Exam
Spring 2018

Student Number: 

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1  2  3  4  5  6  7  8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.
1. We can regard \( \pi_1(X, x_0) \) as base point preserving homotopy classes of maps of \((S^1, pt)\) to \((X, x_0)\). Let \([S^1, X]\) be the set of homotopy classes of maps \(S^1\) to \(X\) (not necessarily base point preserving). There is a natural map

\[ \Psi : \pi_1(X, x_0) \rightarrow [S^1, X] \]

that just ignores the base points. Show that \( \Psi \) is onto if \( X \) is path connected. Also show that \( \Psi([\gamma]) = \Psi([\lambda]) \) if and only if there is some \( g \in \pi_1(X, x_0) \) such that \( [\gamma] = g^{-1} [\lambda] g \).

2. Let \( G \) and \( H \) be groups and \( X \) and \( Y \) topological spaces such that \( \pi_1(X, x_0) \cong G \) and \( \pi_1(Y, y_0) \cong H \). If \( X \) is a finite connected 2-complex, then show that for any homomorphism \( \phi : G \rightarrow H \) there is continuous map \( f : X \rightarrow Y \) such that \( f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0) \) is given by \( \phi \) under the isomorphisms above.

3. Use algebraic topology to show that the free group of rank 2 has a normal subgroup of index 3 and a non-normal subgroup of index 3.

4. Given two points \( x \) and \( y \) in a connected smooth manifold \( M \), show there is a compactly supported isotopy of \( M \) from \( \phi_0 : M \rightarrow M \) to \( \phi_1 : M \rightarrow M \) such that \( \phi_0 \) is the identity on \( M \) and \( \phi_1(x) = y \).

Hint: If helpful, you may assume without proof that in a connected manifold any two points can be connected by a smooth injective path with nonzero derivative at each point.

5. Let \( f : M \rightarrow N \) and \( g : N \rightarrow M \) be two smooth maps between compact oriented manifolds (without boundary) of the same dimension. If \( N \) is connected and \( g \circ f \) is a diffeomorphism show that \( f \) and \( g \) are both diffeomorphisms.

6. Let \( M(n, \mathbb{R}) \) be the set of all \( n \) by \( n \) matrices (recall that it can be identified with \( \mathbb{R}^{n^2} \) by choosing an ordering of the entries of the matrix). Let \( O(n) \) be the subset of \( M(n, \mathbb{R}) \) consisting of matrices satisfying \( A^t A = Id \) where \( A^t \) is the transpose of \( A \) and \( Id \) is the identity matrix. Show that \( O(n) \) is a manifold and compute its dimension. Describe the tangent space to the identity of \( O(n) \).

7. Let \( S_1 \) and \( S_2 \) be two submanifolds of \( M \) (all manifolds are without boundary). Define what it means for \( S_1 \) to be transverse to \( S_2 \) and if they are transverse show carefully that \( S_1 \cap S_2 \) is a submanifold of \( M \) of dimension \( \dim(S_1) + \dim(S_2) - \dim(M) \).

8. Let \( \alpha_1, \ldots, \alpha_k \) be 1-forms on a smooth manifold \( M \) show that they are linearly independent if and only if at some point \( \alpha_1 \wedge \ldots \wedge \alpha_k \neq 0 \).