Property (NL) in Coxeter Groups
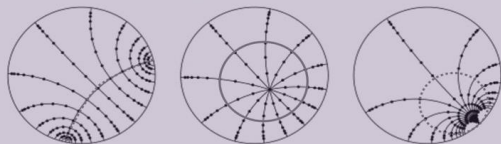
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Property (NL)

Property (NL): describes groups that do not admit a loxodromic action on any hyperbolic space. Stands for no loxodromics.

Loxodromic isometry   Elliptic isometry   Parabolic isometry

Coxeter Groups

Coxeter group presentation:

$$\langle r_1, r_2, \ldots, r_n \mid r_i^2 = (r_i r_j)^{m_{ij}} = 1 \rangle$$

$r_1, r_2, \ldots, r_n$: a set of generating reflections

$(r_i r_j)^{m_{ij}} = 1$: relation between r_i and r_j where $m_{ij} \in \mathbb{N}$

Coxeter graph:

Vertices: represent group generators

Edges: represent relations; labeled m_{ij}

Right-angled Coxeter group: all relations have order 2

D_∞

$x \bullet$
$y \bullet$

$\langle x, y \mid x^2 = y^2 = 1 \rangle$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$\langle a, b, c \mid \begin{matrix} a^2 = b^2 = c^2 = 1 \\ (ab)^2 = (bc)^2 = (ac)^2 = 1 \end{matrix} \rangle$

Theorem: Property (NL) in Right-Angled Coxeter Groups

Let A_Γ, with corresponding graph Γ, be a right-angled Coxeter group. Then,

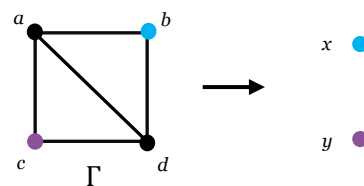$$\Gamma \text{ is complete} \Leftrightarrow A_\Gamma \text{ has Property (NL)}$$

Summary of Proof

Forwards: Γ is complete $\Rightarrow A_\Gamma$ is finite $\Rightarrow A_\Gamma$ has property (NL)

Backwards: Let Γ be an incomplete right-angled Coxeter graph. We define a surjective homomorphism $\varphi: A_\Gamma \to D_\infty$

Lemma 4.2. Let N be a normal subgroup of G and $\varphi: G \to G/N$ be the quotient map. If $G/N \curvearrowright X$ contains a loxodromic element, then so does $G \curvearrowright X$. [1]

Example:

$$A_\Gamma = \langle a, b, c, d \mid \begin{matrix} a^2 = b^2 = c^2 = d^2 = 1 \\ (ab)^2 = (ac)^2 = (ad)^2 = \\ (bd)^2 = (cd)^2 = 1 \end{matrix} \rangle$$

$\varphi: A_\Gamma \to D_\infty$

$\varphi(a) = 1$
$\varphi(b) = x$
$\varphi(c) = y$
$\varphi(d) = 1$

$A_\Gamma / Ker(\varphi) \cong D_\infty$

Triangle Groups

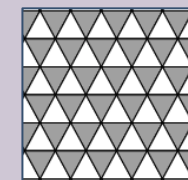For a Coxeter group with three vertices and edge labels $l, m,$ and $n,$

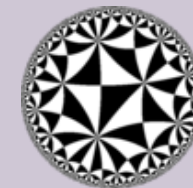$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1 \qquad \frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1 \qquad \frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$$

Spherical (Finite)   Euclidean (Affine)   Hyperbolic

Conjecture: Let A_Γ be a non-right-angled triangle group. Then,

$$A_\Gamma \text{ has property (NL)} \Leftrightarrow A_\Gamma \text{ is finite or affine}$$

Future Research

• Exploring groups generated by more than 3 elements
• Simplifying Coxeter groups though homomorphisms
• Identifying property (NL) in affine Coxeter groups

References

[1] Balasubramanya, S., Fournier-Facio, F., Genevois, A., & Sisto, A. (2022). Property (NL) for group actions on hyperbolic spaces.

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