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## Abstract

First we give bounds on the ratio of the maximum and average size of an independent set fo graphs of high girth. Then we give tight bounds for the difference of the minimum and maximum chromatic number to the average chromatic number over induced subgraphs. Lastly we present an algorithmic way to find the maximum chromatic number given a degree sequence.

## Introduction

Let $G=(V, E)$ be a simple graph with vertex set $V$ and edge set $E$. An induced subgraph, denoted $G[X]$ for some $X \subset V$, is the subgraph with the vertex set $X$ and all edges from $G$ that have both We let $\alpha(G)$ denote the independence number, the size of the largest independent set in $G$.
A proper coloring of $G$ is an assignment of colors to the vertices of $V$ so that no adiacent vertices A proper coloring of $G$ is an assignment of colors to the vertices of $V$ so that no adjacent vertices
are colored the same. Thus a proper coloring is a partitioning of the vertex set into independent sets. Let the chromatic number, denoted $\chi(G)$, be the least number of colors necessary to properly color $G$.
A graph is triangle-free if no three vertices induce three edges. Let the girth of a graph be the length of its shortest cycle. Thus, a triangle-free graph has girth at least 4.


Figure 1. A maximal independent set in red Figure 2. The Moser Spindle properly colored with 4 colors
Average Independence Number
One fairly natural question to consider is whether or not a random independent set from a graph will be close in size to the largest possible independent set. Towards this goal, we make the following definitions:

- $\bar{\alpha}(G)$ is the average size of independent sets in $G$
- An independent set is maximal if it is not contained in any other independent sets.
$\bar{\alpha}_{M}(G)$ is the average size of maximal independent sets in $G$
- For graphs $G$ and $G^{\prime}$, the graph $G+G^{\prime}$ is the graph $G \sqcup G^{\prime}$ with an edge added between every $v \in G$ and $v^{\prime} \in G^{\prime}$

Our question then becomes finding bounds on $r(G)$.
One may compute that $\lim _{n \rightarrow \infty} r\left(K_{n}\right)=1$ and $\lim _{n \rightarrow \infty} r\left(K_{n}+E_{m}\right)=m$, where $E_{m}$ denotes the graph with $m$ vertices and no edges. As such, we impose restrictions on the girth of $G$.
In the triangle-free case, it is conjectured that the lower bound for this ratio is $4 / 3$ (Davies et al. 2017)[1]. Because current lower bounds for $\alpha(G)$ are proven by finding lower bounds for $\bar{\alpha}(G)$ showing this ratio has a lower bound greater than 1 would immediately improve known lower bounds on $\alpha(G)$.
Towards finding an upper bound for $r(G)$, we have shown that $\bar{\alpha}_{M}(G) \leq 2 \bar{\alpha}(G)$, meaning that bounding $\bar{\alpha}_{M}(G)$ from below in terms of $\alpha(G)$ would bound $r(G)$ from above. We conjecture that, if $G$ is of girth at least $5, r(G) \leq 4$. We proved this for the case where no vertices of $G$ share neighbors in any independent set $I \subset G$ such that $|I|=\alpha(G)$.

## Average Chromatic Number

For graph $G=(V, E)$, with $n=|V|$, let $U$ be chosen uniformly at random from the power set of $V$. We define the average chromatic number as

$$
\bar{\chi}(G)=\mathbb{E}[\chi(G[U])]=\frac{1}{2^{n}} \sum_{X \subset V} \chi(G[X])
$$

We consider the minimum and maximum values $\bar{\chi}$.

## For fixed $n$ :

- Minimum: $\bar{\chi} \geq \frac{\chi}{2}$, tight for complete graphs. - Minimum: $\bar{\chi}-\frac{\chi}{2} \geq \frac{1}{2^{n}}$ for non-complete graphs, tight for $K_{n-1} \sqcup K$
- Maximum: $\bar{\chi}-\frac{\chi}{2} \leq \frac{n}{8}$, tight for Turán graphs
with maximal independent sets of size 2 or 3 .
For fixed $\chi$ :
- Maximum: $\bar{\chi}-\frac{\chi}{2} \geq\left(1-o_{n}(1)\right) \frac{\chi}{2}$, ex: Turán Graphs


## Proof Techniques

ments:

- $\chi(G[U])+\chi(G[V \backslash U]) \geq \chi(G)$ for all $U \subset V$. If $G$ is not $k$-partite,
then in any coloring of the vertices there are
two vertices of different color and non-adjacent. - Strong induction on the chromatic number.


## Figure 3. Complete graph on 6 vertices, Turara graphs of maximal independent sets size 2,3



Triangle-free and large girth
Since the above extremal examples are quite dense, a natural next question is for more sparse graphs. For triangle-free graphs we conjecture that the maximum value of $\bar{\chi}-\frac{\chi}{2}$ is

$$
1-2^{\frac{n}{2}+1-n} \text { for } n \text { even and } 1-2^{\frac{n-1}{2}+1-n} \text { for } n \text { odd. }
$$



## Girth at least 5

 For larger girth we conjecture that the best upper bound on $\bar{\chi}-\frac{\chi}{2}$ is $1-o_{n}(1)$, the similar to thetriangle free graphs. Again, below we give the first few unique graphs with maximum $\bar{\chi}-\frac{\chi}{2}$ for girth 5 and 6 .

Maximum Chromatic Number of a Degree Sequence
A degree sequence of a graph $G$ is a non-increasing sequence of the vertex degrees of its vertices. We let $s$ be such a sequence on $n$ vertices, writing $s=v_{1}, \ldots, v_{n}$ where $\operatorname{deg}\left(v_{1}\right) \geq \ldots \geq \operatorname{deg}\left(v_{n}\right)$. We consider how to maximize the chromatic number across all possible graphic configurations of $s$, defining $\chi(s)$ to be the maximum chromatic number that $s$ yields. We first provide the $\chi(s) \leq 1+\max \left(\min \left(i-1, \operatorname{deg}\left(v_{i}\right)\right)[2]\right.$. We then investigate $\chi\left(s_{p}\right)$ the maximum chromatic number of $s$ when we require graphs to be loop-free and connected.

## Alternate Constructive Proof of an Upper Bound on $\chi(s)$

## We will show that $\chi(s)=\mid$ maximal clique of $s \mid$.

From $s$, construct a graph $G$ that has $\chi(G)=k$. Take a critical sub-graph $H$ of $G$. Since $H$ is critical, it has at least $k$ vertices with degree greater than or equal to $k-1$. Take $k$ of these vertices. From these, we are able to form a $k$-clique while maintaining $s$ (omitting the details of how this is done).
Thus, the resulting graph has degree sequence $s$ and chromatic number $k$. Since we can do this for any graph from $s, \chi(s)$ is maximized when the clique is maximized.
This yields the above bound. Let the maximal clique be made of $v_{1}, v_{2}, \ldots, v_{i}$. For $v_{i}$, note that $\min \left(i-1, \operatorname{deg}\left(v_{i}\right)\right)=i-1$; if not, then $i-1<d_{i}$, contradicting it being in the maximal clique. For
$v_{i+1}$, note that $i>d_{i}$, else it would be in the maximal clique. Noting that $i-1$ is the maximum of all $\min \left(i-1, \operatorname{deg}\left(v_{i}\right)\right)$ values and the maximal clique size is $i$, this gives the same bound.

## Requiring Connectednes

Note that the above construction using maximal cliques may result in disconnected, loopNonta dinvestigate how the maximum chromatic number of proper graphs on $s$, termed $\chi\left(s_{p}\right)$, differs from $\chi(s)$.


For cycles of $n$ vertices, each with $n-2$ additional edges connecting to single-degree neighbors, $\chi(s)=$ $n$ while $\chi\left(s_{p}\right)=3$. Thus, $\chi(s)$ can be much larger tha $\chi\left(s_{p}\right)$. To the left, we show the graphs for $n=4$,
However, for this $s$, the average degree $d$ is small $(d=2)$. We are interested in further exploring the following possible conjectures regarding $d, \chi\left(s_{p}\right)$, and $\chi(s)$ :

- For $s$ with average degree $d$ and total number of vertices $n \geq 2 d, \chi\left(s_{s}\right) \geq d$ As $n($ or $d) \rightarrow \infty, \chi\left(s_{p}\right) \rightarrow \chi(s)$


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