Bounding Chromatic Number of Graphs with Girth at Least 5

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Abstract

We give an alternative proof of the known asymptotic upper bound of the chromatic number as a function of the maximum degree on graphs of girth at least 5. We do this by using similar ideas in Noga Alon's proof in [1] giving lower bounds on the independence number.

Introduction

Let G be a graph on vertex set V and edge set E. A **proper coloring** is a coloring of the vertices so that no two adjacent vertices are colored the same. The **chromatic number** of G, denoted $\chi(\mathbf{G})$ is the fewest colors needed to properly color G. An **independent set** I is a subset of the vertices so that no edge in G has both vertices in I. It follows that proper colorings are partitions of the vertices of a graph into independent sets. Let the **independence number** denoted, $\alpha(\mathbf{G})$, be the size of the largest independent set in G. A well-known inequality relating $\chi(G)$ and $\alpha(G)$ is

$$\alpha(G) \cdot \chi(G) \geq |V|.$$

In general, for a graph with maximum degree d, the chromatic number is bounded above by $d+1$ and equality is achieved for graphs containing cliques (and by odd cycles). Thus, a more interesting question is what bounds exist on the chromatic numbers of more sparse graphs. In this instance, we produced upper bounds on the chromatic numbers of graphs where the **girth** (length of shortest cycle) is at least five.

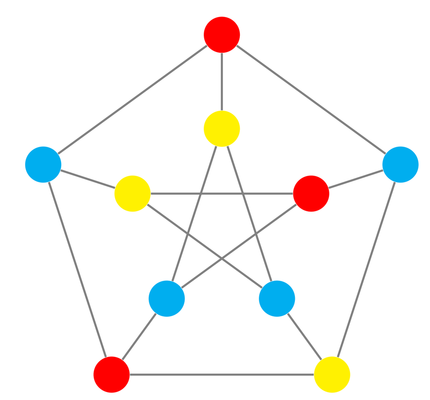


Figure 1. A proper coloring of a graph of girth 5

History

The strongest current methods to bounding the chromatic number or independence number are probabilistic. For a triangle-free graph on n vertices and maximum degree d we present a list of previous results

- $\alpha \geq \Omega\left(\frac{\log d}{d} n\right)$ - Ajtai, Komlós, Szemerédi (1981)
- $\alpha \geq (1 - o_d(1))\left(\frac{\log d}{d} n\right)$ - Shearer (1983)
- $\alpha \geq \Omega\left(\frac{\log d}{d} n\right)$ with a short proof - [1] Alon (1996)

Let $\overline{\alpha}$ denote the average size of an independent set.

- $\overline{\alpha} \geq (1 - o_d(1))\left(\frac{\log d}{d} n\right)$ - [2] Davies, Jensen, Perkins, Roberts (2018)
- $\chi \leq (1 + o_d(1))\left(\frac{d}{\log d}\right)$ - [3] Molloy (2019)

Probabilistic Tools

Chernoff Bound

Let $X_1, \ldots, X_n$ be independent random variables and X to be the sum of the variables X_i. Then for all $\lambda > 0$

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq \lambda\sqrt{n}] \leq e^{-\lambda^2/2}$$

Martingales and Azuma's Inequality

Let $X_0, X_1, \ldots$ be random variables such that for all $i \in \mathbb{N}$

$$\mathbb{E}[X_{i+1} | X_0, \ldots, X_i] = X_i$$

then this sequence of events is called a **martingale**. If $|X_{i+1} - X_i| \leq 1$ for all i, then

$$\mathbb{P}[X_n \geq \lambda\sqrt{n}] \leq e^{-\lambda^2/2}$$

Results

Bounding Expected Size of the Union of Randomly Selected Independent Sets

We can choose a sequence of independent sets $W_1, \ldots, W_k$ iteratively in the following way. First choose an independent set W_1 uniformly at random from the graph G, then choose the second set set W_2 uniformly at random from the graph induced on $V(G) \setminus W_1$, and so on choosing a total of q sets.

Let G be a triangle-free graph with maximum degree d, then there exists a $k \leq O(\frac{d}{\log d})$ such that if $W_1, \ldots, W_k$ is a sequence of independent sets chosen as above, and $W = \cup W_i$ is their union, we have

$$\mathbb{P}[v \in W] \geq 1 - \frac{1}{e^{d^{1/3}}}.$$

Bounding the Chromatic Number of Graphs with Girth at Least 5

For a graph G of girth at least 5 and maximum degree d, $\chi(G) \leq O(\frac{d}{\log d})$.

A Useful Lemma

Our results rely heavily on the following lemma, based on ideas in [1]. The philosophy of its proof is at the core of our project: that when choosing an independent set at random there is a trade off between the probability that a vertex is chosen to be in the set and the expected number of its neighbors which are chosen to be in the set. One of the two of these quantities must be high. Most of our work amounts to choosing the right random variables that witness this trade off.

Let G be a triangle-free graph with maximum degree d. Fix $v \in G$, and choose an independent set W at random. Then, there exist some constants c_1, c_2 such that either

$$\mathbb{P}[v \in W] \geq \frac{c_1}{d^{1/4}}$$

or

$$\mathbb{P}[|N(v) \cap W| \geq c_2 \log d] \geq \frac{1}{4}.$$

We can use this to show that given a graph G triangle-free and maximum degree d, there exist an absolute constant C and a sequence of independent sets $W_1, \ldots, W_q$ such that when $W = \bigcup W_i$ and $q = C\frac{d}{\log d}$, we have the graph induced by $G \setminus W$ has maximum degree $o(d)$.

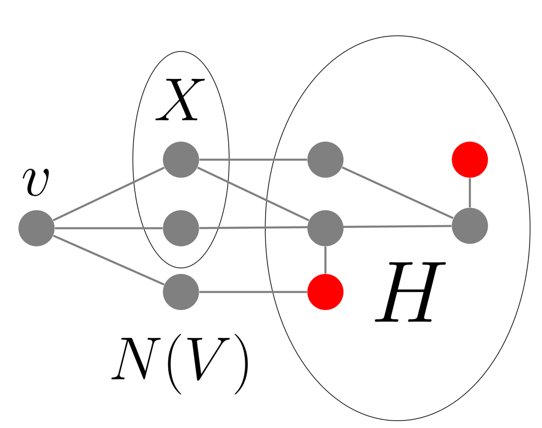


Figure 2. An illustration of the proof

The figure to the left illustrates the idea of the proof of the lemma, due to Noga Alon. Let H be the induced subgraph on $V(G) \setminus (\{v\} \cup N(V))$, and condition on $V(H) \cap W = S$ for some fixed independent set S in H. Then, let X be the set of neighbors of v that are non-neighbors of S, and set $x := |X|$. When x is large, $\mathbb{E}[|N(v) \cap W|]$ is large, and when x is small, we have that $\mathbb{P}[v \in W]$ is large.

Lovász Local Lemma

We use the following version of the Lovász Local Lemma. For each event $X \in \mathcal{B}$ if there exists a d such that there exists a subset $C \subset \mathcal{B}$ with $|\mathcal{B} \setminus C| \leq d$ and that for any subset of events $D \subset C$ we have universal bound on

$$\mathbb{P}[X | \neg D] \leq p,$$

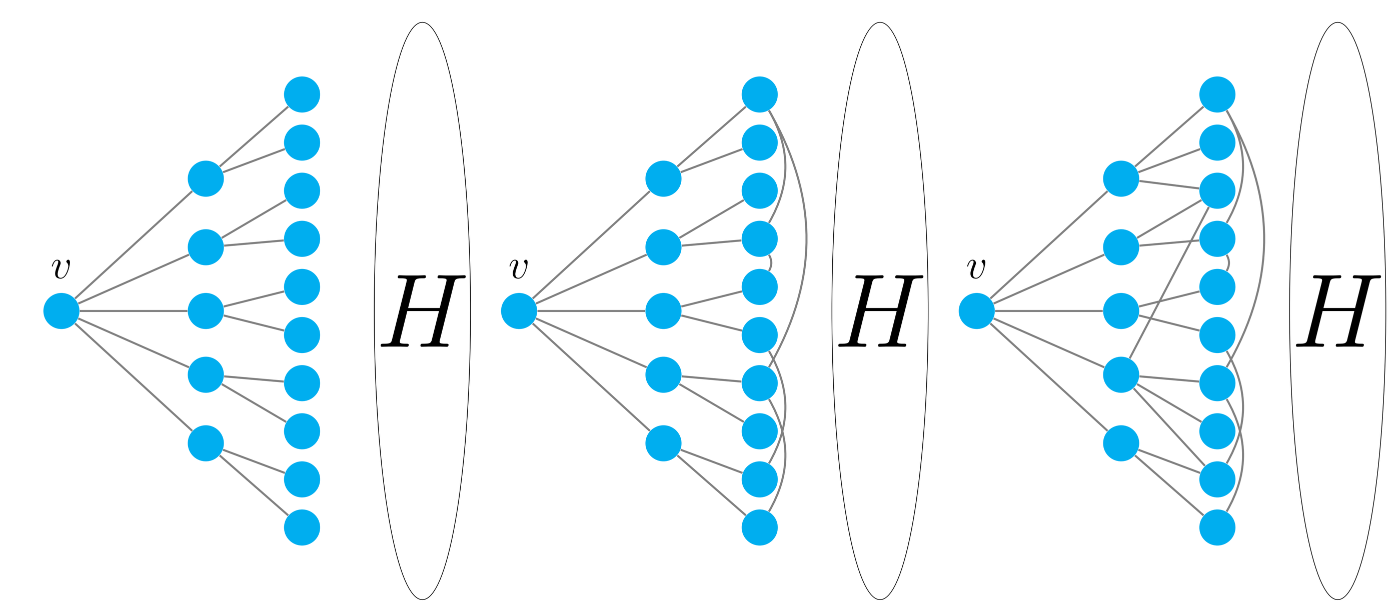
if $ep(d+1) \leq 1$, then no event in $\mathcal{B}$ occurs with positive probability.

Bounding $\chi(G)$

We consider an independent set W chosen uniformly at random. For each vertex $v \in V(G)$, we associate with v the random variable $X_v := \frac{\sqrt[4]{d}}{2}|\{v\} \cap W| + \frac{128}{\log d}|N(v) \cap W|$, and define the random variable

$$Y_v := \sum_{w \in N(v)} X_w.$$

Using a form of the Lovász Local Lemma and a martingale argument, we proved that for graphs G with girth greater than 4, there exists an independent set W such that either $Y_v \geq \frac{d}{8}$ or $v \in W$ for all $v \in V(G)$. Then, we considered a sequence of independent sets $W_1, W_2, \ldots$, with W_i chosen from independent sets on the induced subgraph $G[V(G) \setminus W_1 \cup \cdots W_{i-1}]$ such that either $Y_v \geq \frac{d}{8}$ or $v \in W_i$ for all $v \in V(G) \setminus W_1 \cup \cdots W_{i-1}$. This implies that after taking $q = O(\frac{d}{\log d})$ independent sets, the remaining graph has maximum degree cd for $c < 1$. Considering a geometric series implies that we must only take $O(\frac{d}{\log d})$ independent sets to cover the entire graph, implying that $\chi(G) \leq O\left(\frac{d}{\log d}\right)$, as desired.



The figure above illustrates the neighborhood diagram for Girths 6, 5, and 4 respectively. The structure of this diagram in each of the three cases determines our ability to bound the probability that the random variable Y_v is sufficiently large most of the time.

Future directions

Our primary focus for future study is removing the assumption that the graph G is girth-5 from the argument that bounds chromatic number. We have evidence that the bounds we have established for our random variables remain true in the girth-4 case and we believe that proving so is within the reach of our current methods. Lastly we hope to using some ideas from [2] to improve the coefficient on the upper bound of $\chi(G)$ to $(1 + o_d(1))$, the known best upper bound [3].

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