Concordances of linear combinations of Torus knots to L-space knots

Dan Guyer, Thomas Sachen, Advisor: Dr. JungHwan Park

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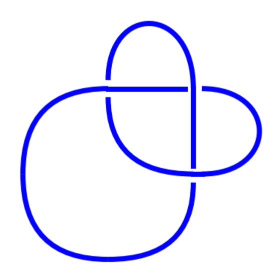
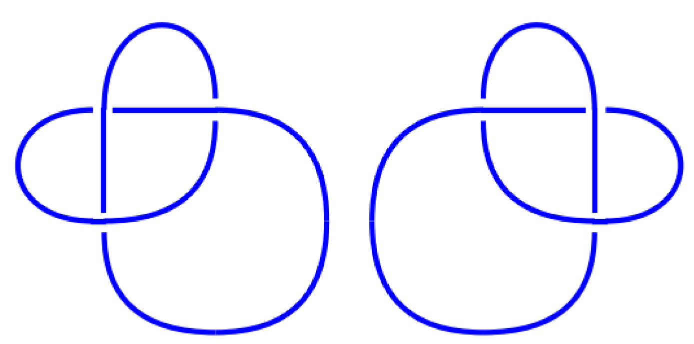
Background

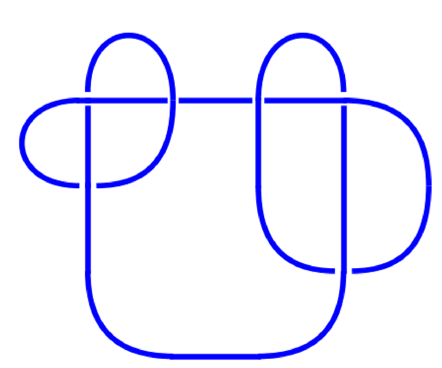
- **Knot:** An embedding of S^1 into $\mathbb{R}^3$ (or S^3, its one-point compactification).



- **Torus Knot ($T(p,q)$) :** A knot that can be drawn on a torus T, *i.e.*, an embedding $S^1 \to T$. These knots are uniquely determined (up to isotopy) by coprime integers p and q, where $T(p,q)$ winds around the torus p times longitudinally (around the donut hole) and q times meridionally (through the donut hole).
- **Mirror Operation:** An operation on a knot that produces a knot as if it was reflected about a mirror, denoted by $-K$ for a knot K.



- **Connected Sum:** An operation on two knots that cuts a strand from each knot to then join the two knots together, denoted by $K\#J$ for two knots K, J.



- **L-space:** a $\mathbb{Q}$-homology 3-sphere with the "simplest" possible Heegaard-Floer homology, *i.e.* for which $|H_1(Y;\mathbb{Z})| = \operatorname{rank} \widehat{HF}(Y)$. Includes lens spaces.
- **L-space knots:** Knots that admit positive Dehn surgeries to L-spaces. Includes all positive torus knots.
- **Knot Invariant:** A function with domain as the set of (isotopy classes of) all knots.

  If an invariant gives two distinct values for two knots, then the knots are distinct.

  – Examples:
    - ∗ **Alexander Polynomial:** A symmetric polynomial in t, denoted by $\Delta_K(t)$ for a knot K.
    - ∗ **Determinant:** A knot invariant that associates an integer to each knot, denoted by $\det(K)$. Furthermore, $\det(K) = \Delta_K(-1)$.
    - ∗ **Signature:** A knot invariant that associates an even number to each knot, denoted by $\sigma(K)$.

The Knot Concordance Group

- **Slice:** A knot is *slice* if it bounds a smooth disk in B^4 (the 4-D ball).
- **Knot Concordance:** Two knots K, J are *concordant* if $K\# - J$ is slice.
- **Knot Concordance Group ($\mathcal{C}$):** The group of concordance classes of knots with the operation of connected sum.
  - **Associativity:** Follows easily from the connected sum operation.
  - **Identity:** The class of all slice knots.
  - **Inverses:** Given a knot K, its inverse in $\mathcal{C}$ is $-K$, the mirror image of K.

$$\mathcal{C} \twoheadrightarrow \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$$

- **Concordance Invariant:** A function with domain as the set of concordance classes of knots.

  If a concordance invariant gives two distinct values for two knots, then the knots are not concordant.

  – Examples:
    - ∗ **Tau:** An integer valued concordance invariant, denoted by $\tau(K)$ for some knot K.
    - ∗ **Upsilon:** A piecewise linear function defined over the interval $[0,2]$, denoted by $\Upsilon_K(t)$.
    - ∗ **Signature:** A knot invariant that associates an even number to each knot, denoted by $\sigma(K)$. (As previously described.)

Previous Work

- C. Livingston, [3]
  - *Theorem 1.1 - Let $\{p_i, q_i\}_{i=1,\dots,n}$ be a set of pairs of relatively prime positive integers with $2 \leq p_i < q_i$ for all i and with $n > 1$. Then $\#_i T(p_i, q_i)$ is not concordant to an L-space knot.*
- S. Allen, [2]
  - *Conjecture 1.2 If a linear combination of torus knots is concordant to an L-space knot, then it is concordant to a positive torus knot.*
  - *Theorem 1.1 - If the connected sum of distinct positive torus knots $mT(p,q)\#nT(r,s)$ is concordant to an L-space knot, then either $m = 0$ and $n = 1$ or $m = 1$ and $n = 0$.*
  - *Proposition 4.1 - If the knot $K = T(p_1,q_1)\#T(p_2,q_2)\#\dots\#T(p_m,q_m)\# - T(p_1',q_1')\#\dots\dots - T(p_n',q_n')$ where $m, n \geq 1$ is concordant to an L-space knot J, then*

$$\frac{\prod_{i=1}^m \Delta_{T(p_i,q_i)}(t)}{\prod_{i=1}^n \Delta_{T(p_i',q_i')}(t)} = \Delta_J(t)$$

- Aceto-Celoria-Park, [1]
  - *Corollary 1.7 - Any smooth concordance class in the subgroup generated by 2-bridge knots is represented by a connected sum of 2-bridge knots K such that if J is concordant to K, then $\det(K)$ divides $\det(J)$. Moreover, as a connected sum of 2-bridge knots K is uniquely determined up to isotopy.*

New Results

Theorem 1. *Let K be a nontrivial linear combination of $T(2,q_i)$ torus knots. Then K is not concordant to an L-space knot.*

- **Proof Sketch:**
  - Suppose K is concordant to an L-space knot J
  - Compute $\det(K)$.
  - Compute $\det(J)$ using Proposition 4.1 of [2].
  - From Corollary 1.7 of [1], we find that $\det(K) | \det(J)$.
  - This reduces to the question answered by Livingston in [3].

Theorem 2. *Let K be a nontrivial linear combination of torus knots. Suppose there exists a $-T(p_m,q_m)$ such that $p_m > p_i$ for all $i \neq m$. Then K is not concordant to an L-space knot.*

- **Proof Sketch:**
  - Suppose K be concordant to an L-space knot J and that there exists a $-T(p_m,q_m)$ such that $p_m > p_i$ for all $i \neq m$.
  - By analyzing the Upsilon invariant, $\Upsilon_K(t)$, we find that $\Upsilon_K'(t)$ "jumps" first at $\frac{2}{p_m}$.
  - At $\frac{2}{p_m}$, we have that $\Upsilon_K'(t)$ is decreasing.
  - Hence K cannot be concordant to J since $\Upsilon_J'(t)$ is increasing for all L-space knots.

Further Directions

- **Within Conjecture 1.2 of [2]:**
  - Examine linear combinations of torus knots with a fixed p for all torus knots in our connected sum.
  - Examine linear combinations of torus knots with odd p and q for all p, q.
  - Examine linear combinations of torus knots with p being even for all p.
- **Further questions:**
  - Can any nontrivial connected sum of L-space knots be concordant to an L-space knot?
  - Can any nontrivial linear combination of L-space knots be concordant to an L-space knot?

Acknowledgements

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[1] Paolo Aceto, Daniele Celoria, and JungHwan Park. *Rational cobordisms and integral homology*. 2018. arXiv: 1811.01433 [math.GT].

[2] Samantha Allen. "Concordances from differences of torus knots to L-space knots". In: *Proc. Amer. Math. Soc.* 148.4 (2020), pp. 1815–1827. ISSN: 0002-9939. DOI: 10.1090/proc/14833. URL: https://doi.org/10.1090/proc/14833.

[3] Charles Livingston. "Concordances from connected sums of torus knots to L-space knots". In: *New York J. Math.* 24 (2018), pp. 233–239. URL: http://nyjm.albany.edu:8000/j/2018/24_