

Geodesic-preserving bijections of Thurston geometries Matthew Palani Lideros¹, Akash Narayanan² Mentors: Ryan Dickmann², Dan Margalit² ¹University of California, Berkeley, ²Georgia Institute of Technology

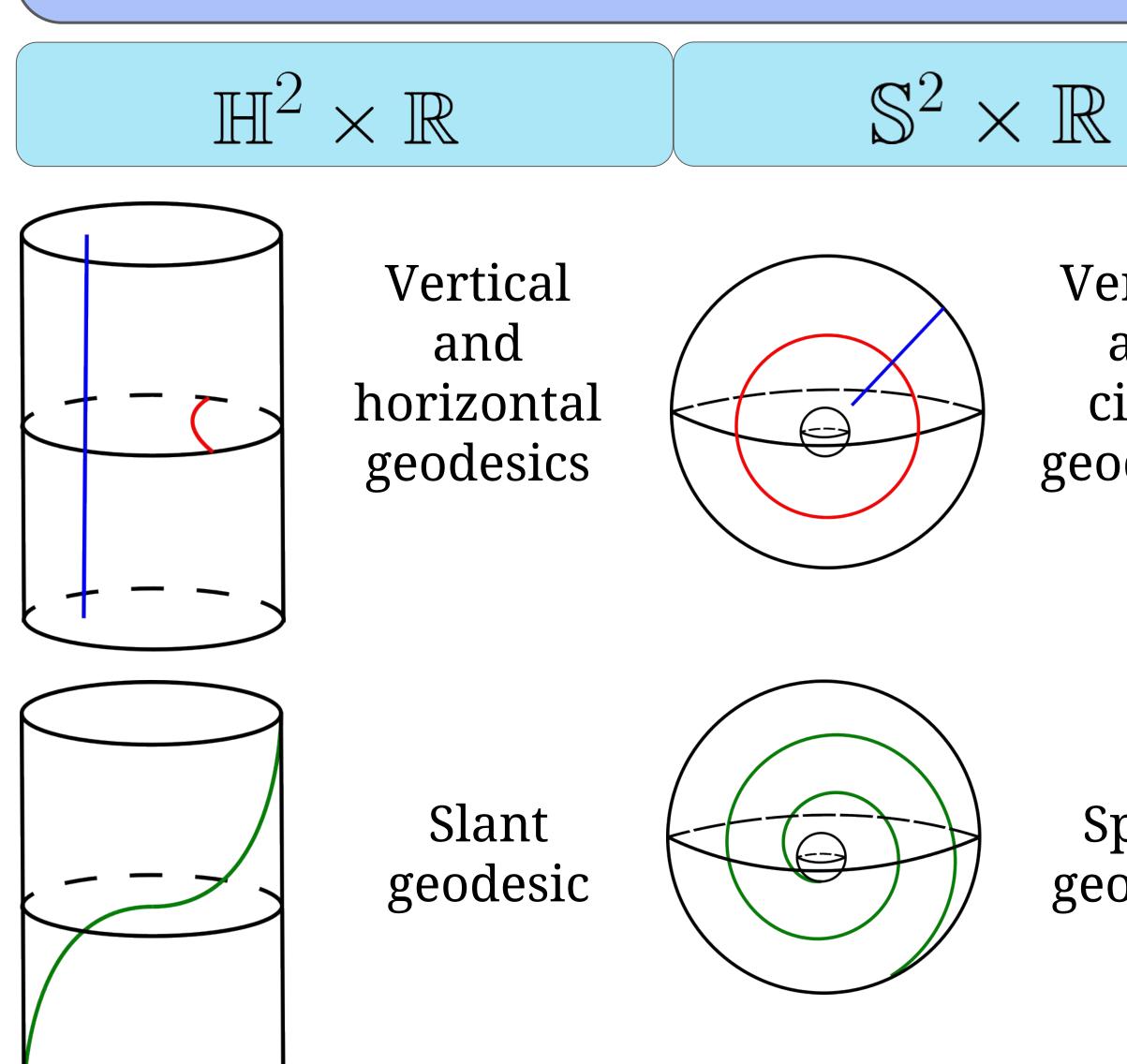
Classical Reconstruction Theorems

bijection sending lines to lines. Then f is an affine transformation.

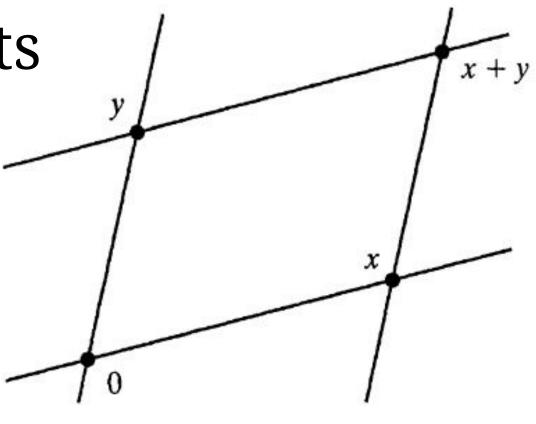
- 1. *f* sends intersection points to intersection points
- sends parallelograms to parallelograms
- **3.** f(x + y) = f(x) + f(y)
- **4.** f(xy) = f(x)f(y)
- 5. *f* is the identity

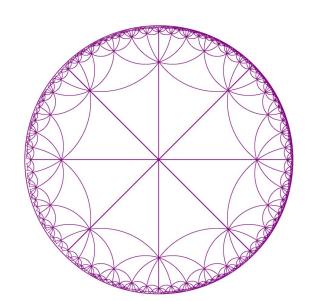
Jeffers extended this to the hyperbolic and spherical cases:

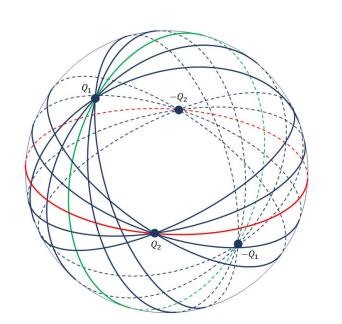
Question: Can we classify geodesic-preserving bijections of the eight Thurston geometries?

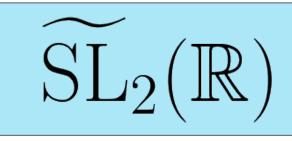


Theorem (Von Staudt, 1847): Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a

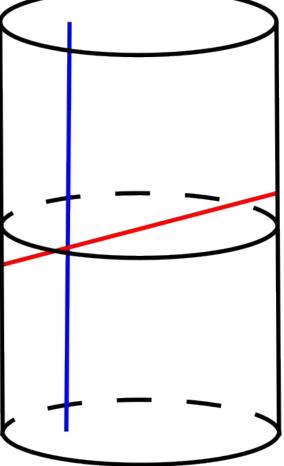








Vertical and circle geodesics



and horizontal geodesics

Oblique geodesic

Vertical

Spiral geodesic

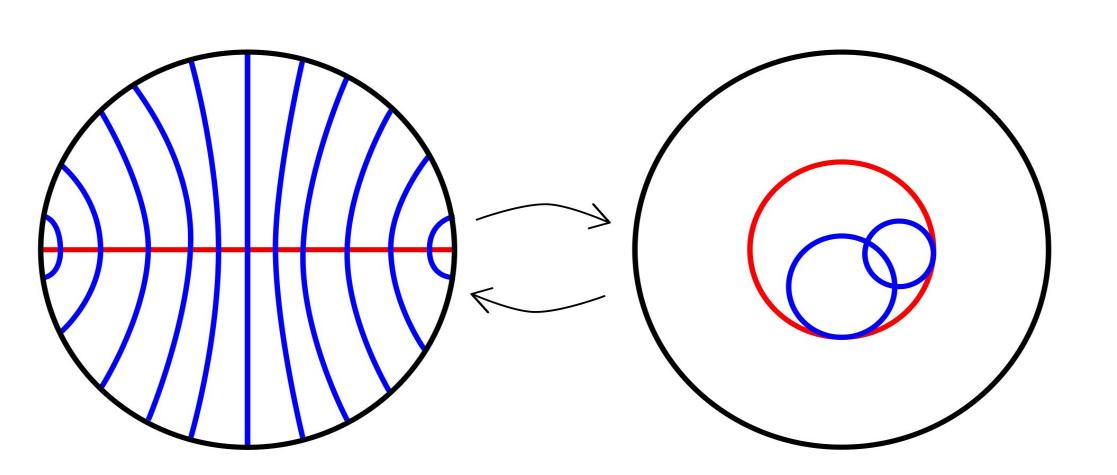
Main Results

Geodesic-preserving bijections of the eight Thurston geometries

X	Group of geodesic-preserving bijections	Status
\mathbb{R}^3	$\operatorname{Aff}(\mathbb{R}^3)$	Von Staudt (1847)
\mathbb{H}^3	$\operatorname{Isom}(\mathbb{H}^3)$	Jeffers (2000)
\mathbb{S}^3	$(\mathbb{Z}/2\mathbb{Z})^{\infty} \rtimes \operatorname{Isom}(\mathbb{S}^3)$	Jeners (2000)
$\mathbb{H}^2\times\mathbb{R}$	$\operatorname{Isom}(\mathbb{H}^2 \times \mathbb{R}) \rtimes \mathbb{R}^+$	Dickmann-
$\mathbb{S}^2 \times \mathbb{R}$	$\operatorname{Isom}(\mathbb{S}^2 \times \mathbb{R}) \rtimes \mathbb{R}^+$	Lideros-
$\widetilde{\mathrm{SL}}_2(\mathbb{R})$		Narayanan
$SL_2(\mathbb{K})$	$\operatorname{Isom}(\operatorname{SL}_2(\mathbb{R}))$	(2023)
Nil	$\operatorname{Isom}(\operatorname{Nil})$	In-progress
Sol	$\operatorname{Isom}(\operatorname{Sol})$	III-progress

Proof Technique: Intersection Patterns

Non-example of geodesic-preserving map in $SL_2(\mathbb{R})$ (top view)



If a map is a geodesic-preserving bijection, then intersection points of geodesics are preserved.

> Acknowledgements We would like to thank the NSF and Georgia Tech.



