The Eigenvector-eigenvalue Identity: Generalizations and Extensions

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The Eigenvector-Eigenvalue Identity

In 2019, while studying neutrino oscillation probabilities, three physicists, Denton, Parke, and Zhang, discovered a formula for 3×3 Hermitian matrices that related their eigenvectors to their eigenvalues. By using this formula, they were able to calculate mixing angles in matter given eigenvalues from the neutrino oscillation Hamiltonian. By using approximate eigenvalues, they were able to compute mixing methods significantly faster than other methods, and with improved accuracy. The team was also able to generalize their formula for 3×3 Hermitian matrices to $n \times n$ Hermitian matrices. They called this expression the eigenvector-eigenvalue identity.

Eigenvector-Eigenvalue Identity

For any diagonalizable $n \times n$ matrix M, with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$, where $v_{i,j}$ is the j^{th} entry of the normalized eigenvector corresponding to eigenvalue λ_i, and M_j is the principal j^{th} minor of M with eigenvalues $\mu_{1,j} \leq \ldots \leq \mu_{n-1,j}$,

$$|v_{i,j}|^2 = \frac{\Pi_{k=1}^{n-1}(\lambda_i - \mu_{k,j})}{\Pi_{k=1;k\neq i}^{n}(\lambda_i - \lambda_k)}.$$

Survey

After contacting Tao regarding their result, they discovered that the eigenvector-eigenvalue identity was not well-known to the mathematical community. However, a survey showed that the first precursor was published by Jacobi in 1834, and it was found independently in over a dozen papers. It is believed that since most authors used the eigenvector-eigenvalue identity as only a step in their research papers, they didn't assign it a name, and it typically appeared in an equivalent but less clearly useful form, the eigenvector-eigenvalue identity was not well known until Denton, Parke, Zhang, and Tao published a survey on the subject in 2021 called "Eigenvectors from Eigenvalues: A Survey of a Basic Identity in Linear Algebra".

Definitions

A real-valued, continuous-time stochastic process B_t, $t \geq 0$ is a **Brownian motion** if

1. $B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent
2. For any nonnegative s, t, $B_t - B_s$ has normal distribution $(0, t-s)$.
3. The map taking t to B_t is continuous with probability one

A **GUE** (Gaussian Unitary Ensemble), consists of $n \times n$ Hermitian matrices with iid diagonal entries with distribution $N(0,1)_{\mathbb{R}}$ and iid non-diagonal entries with distribution $N(0,1)_{\mathbb{C}}$.

Connections with Random Matrix Theory

In 2001, Baryshnikov published a paper containing a form of the eigenvector-eigenvalue identity that he independently discovered and utilized as a lemma in a proof concerning the equivalence in law between the process

$$D_k = \sup\left\{\sum_{i=1}^{k}[B_i(t_i) - B_i(t_{i-1})], 0 = t_0 \leq t_1 \leq \ldots \leq t_k = 1\right\},$$

and the maximal eigenvalue of the principal minors of an infinite matrix in the GUE, where $k \in \mathbb{N} \cup 0$, the B_i are standard Brownian motions. It is useful to show the equivalence between Baryshnikov's results and the eigenvector-eigenvalue identity.

Let $W \cong \mathbb{C}^M$ be a standard complex Hilbert space with norm $\Sigma x_i \bar{x}_i$. Let A be an M-dimensional Hermitian form on W, with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_M$, and let $L^{M-1} \in \mathbb{C}^M$ be the hyperplane defined by $L = \{q \in \mathbb{C}^M : \langle \ell, q \rangle = 0\}$, where $\ell \in \mathbb{C}^M$. Now, consider the restriction $A|_L$ of the Hermitian form A on L. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_{M-1})$ be the set of eigenvalues of $A|_L$ such that $\mu_j \leq \mu_{j+1}$ for integer $1 \leq j < M-1$. Baryshnikov shows that $|\ell_i|^2$ has a uniform distribution in a simplex Σ if l is uniformly distributed in a unit sphere

More on Connections with Random Matrix Theory

This relation allows us to set up a system of equations to solve, which yields

$$|\ell_i|^2 = \frac{\Pi_{1\leq j\leq M-1}(\lambda_i - \mu_j)}{\Pi_{1\leq j\leq M; j\neq i}(\lambda_i - \lambda_j)},$$

Let ℓ be a standard basis vector e_j. Then, $v_{i,j} = \ell_i$. Substituting this into our previous expression gives

$$|v_{i,j}|^2 = \frac{\Pi_{k=1}^{n-1}(\lambda_i - \mu_{k,j})}{\Pi_{k=1;k\neq i}^{n}(\lambda_i - \lambda_k)},$$

which is the eigenvector-eigenvalue identity.

Generalizations and Connections

The work of Baryshnikov is extended by Benaych-Georges and Houdré from an equality in law involving just maximal eigenvalues to the full spectrum of the principal minors of the matrix. This result is presented in Theorem 2.1, which states that

$$\left(\sum_{i=1}^{\ell} \mu_i^k\right)_{1\leq \ell\leq k\leq M} \overset{\text{law}}{=} \gamma(\ell, k) :=$$

$$\left(\sup\left\{\Sigma_{i=1}^{\ell} \Delta_{\pi_i}(B)\,;\, \pi_1, \ldots, \pi_\ell \in \mathcal{P}, \pi_1 < \ldots < \pi_\ell \leq k\right\}\right)_{1\leq \ell\leq k\leq M}.$$

In particular, note that if we let $\ell = 1$, we recover Baryshnikov's result. By using this generalization, it is possible to extend the eigenvector-eigenvalue identity to the full spectrum of the principal minors the matrix. This is useful for two reasons - it allows us to extend Theorem 2.1 to the case where the diagonal vector has covariance Σ, and it also allows us to improve the speed with which neutrino oscillation probabilities are calculated by an order of magnitude.

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