

Introduction

We present **novel combinatorial formulas** for the weighted Ehrhart series of order polytopes, which are polytopes defined from the relations of a partially-ordered set (poset).

- Generalizes work of Richard Stanley in [3][4] in unweighted case
- Encodes information about geometry of order polytope **solely from information about associated poset**
- Allows for **significantly faster computation** of weighted Ehrhart series on order polytopes of large posets than current fastest open-source algorithms
- Yields partial results on real-rootedness of weighted h^* -polynomial of order polytopes

Weighted Ehrhart Series and h^* -Polynomials

Given a polytope $\mathcal{P} \subseteq \mathbb{R}^n$ and polynomial weight function $w : \mathbb{R}^n \rightarrow \mathbb{R}$, the **weighted Ehrhart series** counts the weighted sum of integer points in dilates of $\mathcal{P}$:

$$\text{Ehr}_{\mathcal{P},w}(z) := \sum_{t \geq 0} \left(\sum_{\mathbf{x} \in \mathcal{P} \cap \mathbb{Z}^n} w(\mathbf{x}) \right) z^t$$

If $w = 1$, we get the **unweighted Ehrhart series**, which counts the number of integer points in dilates of $\mathcal{P}$.

By [1], $\text{Ehr}_{\mathcal{P},w}(z)$ can be written as a rational function $h_{\mathcal{P},w}^*(z)/(1-z)^{n+m+1}$, where $h_{\mathcal{P},w}^*(z)$ is the **weighted h^* -polynomial** of $\mathcal{P}$ of degree $\leq n + m$.

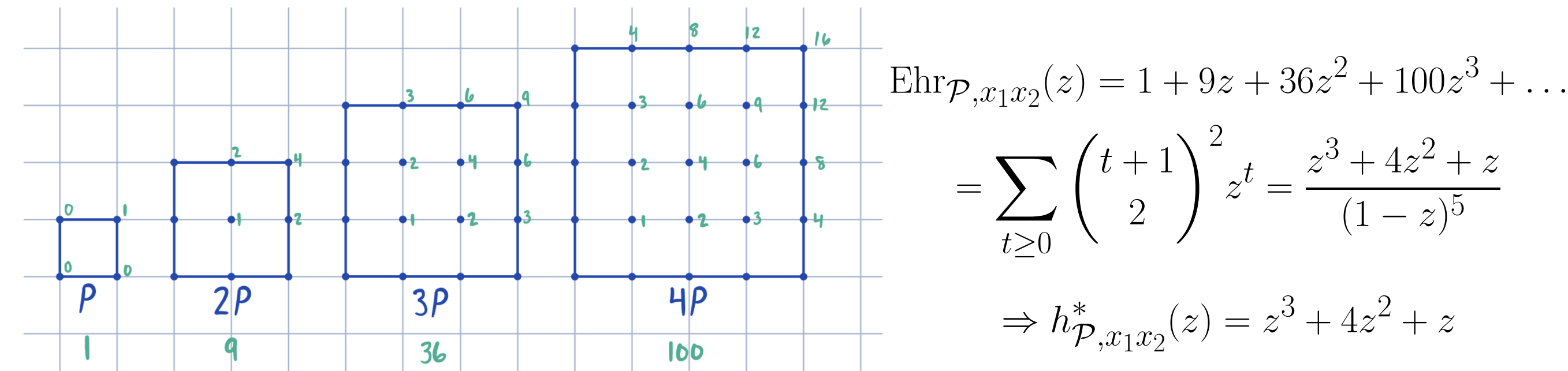


Figure 1. Computing the weighted Ehrhart series and h^* -polynomial for $\mathcal{P} = [0, 1]^2$ and $w(x_1, x_2) = x_1x_2$.

Both the weighted Ehrhart series and h^* -polynomial encode geometric information about the polytope. For example, one can think of the weighted sum over integer points as a discrete analogue of the weighted volume integral.

Order Polytopes

Given a poset P on $[n]$, define the **order polytope** in $\mathbb{R}^n$ by slicing the unit cube with hyperplanes defined by the order relations of P :

$$\mathcal{O}(P) := \{\mathbf{x} \in [0, 1]^n \mid x_i \leq x_j \text{ when } i \preceq_P j\}.$$

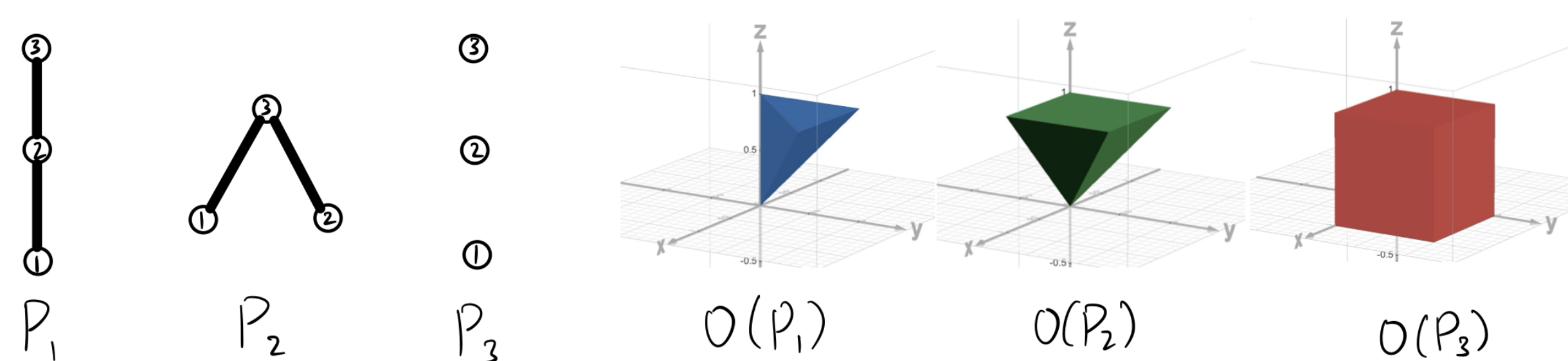


Figure 2. Three posets on [3] and their order polytopes.

In the unweighted case, Stanley gives a purely combinatorial formula for the Ehrhart series of an order polytope:

Theorem ([4], Theorem 3.15.8)

$$\text{Ehr}_{\mathcal{O}(P),1}(z) = \sum_{\pi \in \mathcal{L}(P)} \frac{z^{d(\pi)}}{(1-z)^{n+1}}$$

where $\mathcal{L}(P)$ is the set of linear extensions of P and $d(\pi)$ is the descent number of π .

Here, we generalize his work to the weighted case.

Multivariate Ehrhart Series of Order Polytopes

Given a naturally-labeled poset P on $[n]$, let $\mathcal{T}_m$ be the set of order-preserving maps

$$\mathcal{T}_m = \{\tau : P \rightarrow \{0, 1, \dots, m\} \mid i \leq_P j \Rightarrow \tau(i) \leq \tau(j)\}$$

and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ be formal variables. Then the **multivariate Ehrhart series** is given by

$$\text{Ehr}_{\mathcal{O}(P),\mathbf{y}}(z) := \sum_{m \geq 0} \sum_{\tau \in \mathcal{T}_m} \left(\prod_{j=1}^n y_j^{\tau(j)} \right) z^m.$$

Note the correspondence $(x_1, \dots, x_n) \longleftrightarrow (\tau(1), \dots, \tau(n))$ between coordinates and evaluations of order-preserving maps. Our first result expresses this series using only poset statistics, summing over the linear extensions π of P .

Theorem 1. [H.-J.-Z.]

$$\text{Ehr}_{\mathcal{O}(P),\mathbf{y}}(z) = \frac{1}{1-z} \sum_{\pi \in \mathcal{L}(P)} z^{d(\pi)} \left(\prod_{j=1}^n y_j^{b_j(\pi)} \right) \left(\prod_{k=1}^n \frac{1}{1 - W_k(\pi)z} \right),$$

where $b_j(\pi)$ is the number of descents in π before j and $W_k(\pi) = \prod_{j \in P \mid \pi^{-1}(j) \geq k} y_j$.

Specializing to Weighted Ehrhart Series on Order Polytopes

One of the reasons that the multivariate Ehrhart series is important is that it can be specialized to give the weighted Ehrhart series for any polynomial weight. Given a multiset S of $[n]$, we can associate a monomial weight $w(x_1, \dots, x_n) = \prod_{s \in S} x_s$, for which

$$\text{Ehr}_{\mathcal{O}(P),w}(z) = \left[\left(\prod_{s \in S} y_s \frac{\partial}{\partial y_s} \right) \text{Ehr}_{\mathcal{O}(P),\mathbf{y}}(z) \right] \Big|_{\mathbf{y}=1}.$$

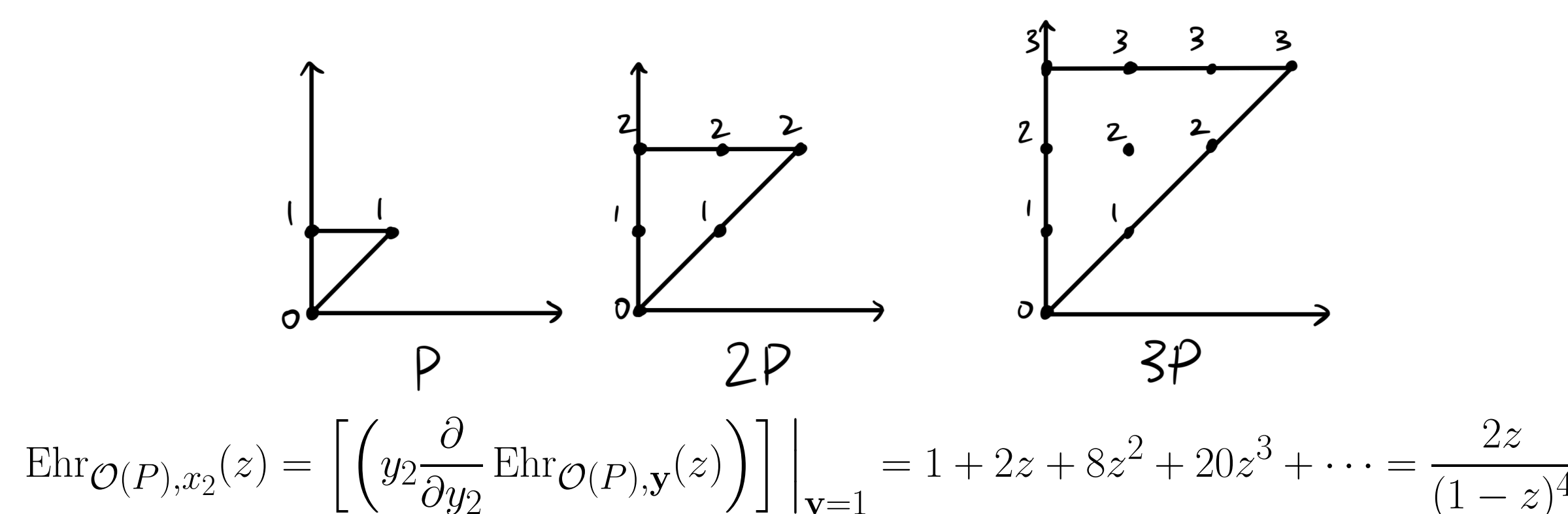
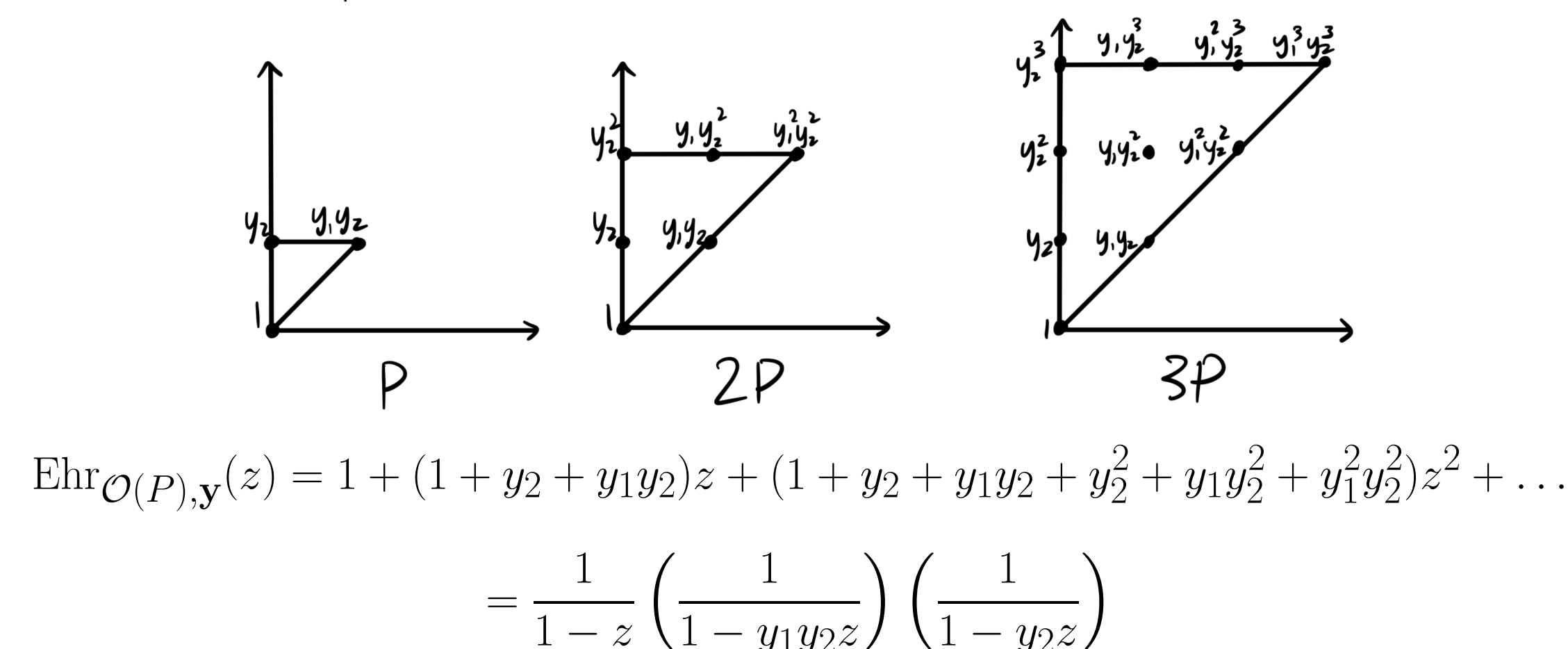
Theorem 2. [H.-J.-Z.] Call $\phi : S \rightarrow \{0, 1, \dots, n\}$ a *valid assignment* for π if $\phi(i) = k > 0 \Rightarrow \pi^{-1}(i) \geq k$. Then

$$\text{Ehr}_{\mathcal{O}(P),w}(z) = \frac{1}{1-z} \sum_{\pi \in \mathcal{L}(P)} \frac{z^{d(\pi)}}{(1-z)^n} \sum_{\text{valid } \phi} \left(\prod_{i \in \phi^{-1}(0)} b_i(\pi) \right) \left(\prod_{k=1}^n P_{|\phi^{-1}(k)|} \left(\frac{z}{1-z} \right) \right),$$

where the polynomials P_d are the Fubini polynomials $P_d(z) = \sum_{j=0}^d j! S(d, j) z^j$ and $S(d, j)$ are the Stirling numbers of the second kind.

The appearance of the Fubini polynomials is quite nice, as they have interlacing roots and thus provide a possible method for proving real-rootedness of the weighted h^* -polynomial.

Example: multivariate Ehrhart series and specialization to weight $w(x_1, x_2) = x_2$ for the order polytope of the 2-chain poset.



Alternatively, a colored stars-and-bars argument yields the following formula, which is faster to compute with. Fix an indexing of the weight $w_\pi(\mathbf{x}) = x_{i_1}x_{i_2} \dots x_{i_s}$ that respects $\pi^{-1}(i_1) \leq \pi^{-1}(i_2) \leq \dots \leq \pi^{-1}(i_s)$. Then define $S = [s]$ and $r(j, S_0) = \#\{\ell \in S_0 : \ell < j\}$.

Theorem 3. [H.-J.-Z.] We can alternatively express $\text{Ehr}_{\mathcal{O}(P),w}(z)$ as

$$\sum_{\pi \in \mathcal{L}(P)} \frac{z^{d(\pi)}}{(1-z)^{n+1}} \sum_{S_0 \subseteq S} \left(\frac{z}{1-z} \right)^{|S_0|} \left(\prod_{j \in S_0} (\pi^{-1}(i_j) + r(j, S_0)) \right) \left(\prod_{j \in S \setminus S_0} (b_{i_j}(\pi) + r(j, S_0)) \right).$$

From either formula for the weighted Ehrhart series, we can recover a combinatorial formula for the weighted integral over the order polytope:

$$\int_{\mathcal{O}(P)} w(\mathbf{x}) d\mathbf{x} = \frac{1}{(d+n)!} \sum_{\pi \in \mathcal{L}(P)} \pi^{-1}(i_1)(\pi^{-1}(i_2) + 1) \dots (\pi^{-1}(i_n) + n - 1).$$

Computational Comparison

Implementing Theorem 3 in SageMath yields an algorithm for computing the weighted Ehrhart series of order polytopes with runtime

$$O(|\mathcal{L}(P)| \cdot 2^s \cdot \text{poly}(ns)) \leq O(n! \cdot 2^s \cdot \text{poly}(ns))$$

where n is the size of the poset and s is the degree of the monomial weight. Below, we compare our algorithm against Normaliz's WeightedEhrhartSeries function, the fastest known open-source algorithm.

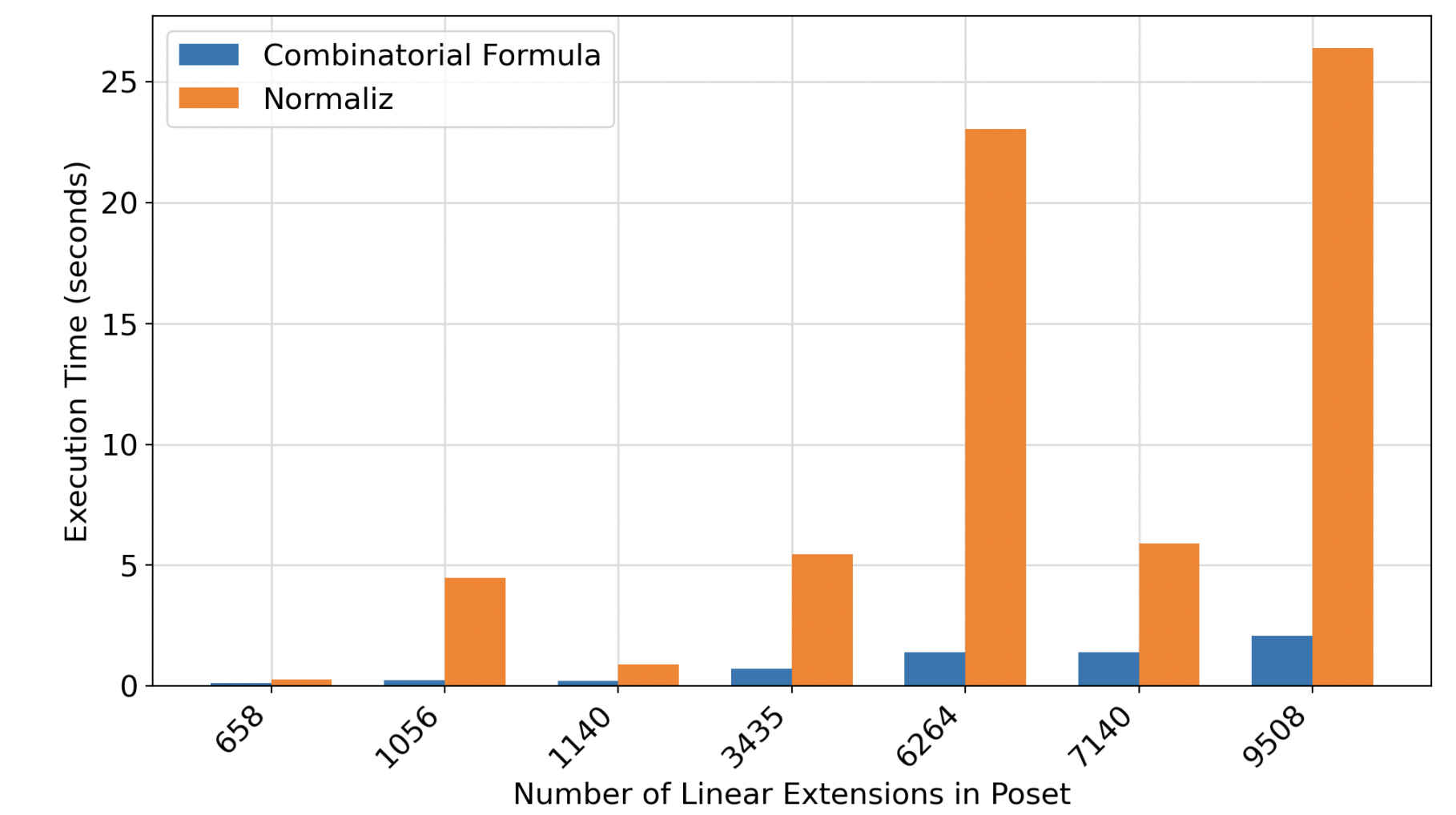


Figure 3. Runtime in seconds of our algorithm compared to Normaliz on 7 randomly-generated (poset, weight) pairs with $n = 15$ and $s = 5$.

Results on Real-Rootedness

We are interested in whether the h^* -polynomial has all real roots, as this would imply properties about the shape of the polynomial's coefficient vector. The Stembridge poset is a counterexample to h^* -polynomial real-rootedness on order polytopes in the unweighted case.

Goal: make progress towards classifying the pairs (poset, monomial weight) that yield weighted h^* -real-rootedness of the order polytope.

Current results:

- Let P be the Stembridge poset and $\bar{P}$ its disjoint sum with a single element. Then $(\mathcal{O}(\bar{P}), 1)$ is real-rooted while $(\mathcal{O}(\bar{P}), x_1)$ is not.
- For any chain C_n , $(\mathcal{O}(C_n), w)$ for any monomial weight is real-rooted. While this is known, we found two new simple proofs directly obtained from our formulas.

Conjecture. Let P be a poset. For squarefree weights w consisting only of elements of P comparable to all other elements, then $(\mathcal{O}(P), w)$ is real-rooted.

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References

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