**Introduction**

When a link \( L \) has non-zero determinant, the maximum Euler characteristic of a surface with \( L \) as its boundary is 1. However, much is still unknown about which links bound surfaces of maximal Euler characteristic in the 4-ball.

The goal of our project is to characterize which four-stranded pretzel links with non-zero determinant bound a maximal Euler characteristic surface in \( B^4 \). We call these links \( \chi \)-slice.

We obstruct most 4-stranded pretzel links from being \( \chi \)-slice using Donaldson’s Diagonalization Theorem. We then prove many of the remaining links are \( \chi \)-slice.

**Fact:** Associated to any oriented 4-manifold \( X \) is a symmetric, integer \( n \times n \) matrix called the intersection form, denoted by \( Q_X \). This matrix is definite if it has all positive or all negative eigenvalues.

**Donaldson’s Diagonalization Theorem:** \([2]\) The intersection form of a closed, definite, oriented, smooth 4-manifold is diagonalizable over \( \mathbb{Z} \).

A consequence of Donaldson’s theorem is that there exists a lattice embedding \( (\mathbb{Z}^n, Q_X) \to (\mathbb{Z}^r, \pm I) \) when \( X \) is a closed, definite, oriented, smooth 4-manifold.

A lattice is a pair \((\mathbb{Z}^n, Q)\), where \( Q \) is a symmetric matrix with non-zero determinant. Let \((\mathbb{Z}^n, Q_1)\) and \((\mathbb{Z}^n, Q_2)\) be lattices. A map \( \phi : \mathbb{Z}^n \to \mathbb{Z}^n \) satisfying

\[
\phi(v + w) = \phi(v) + \phi(w)
\]

\[
\phi(v^T w) = \phi(v)^T Q \phi(w)
\]

for all \( v, w \in \mathbb{Z}^n \) is called a lattice embedding. We write \( \phi : (\mathbb{Z}^n, Q_1) \to (\mathbb{Z}^r, Q_2) \).

**Theorem:** \([1, 2, 5]\) Let \( L \subseteq S^3 \) be a link with \( \Delta(L) \neq 0 \). Suppose that \( L \) is \( \chi \)-slice and that \( \Sigma_2(L) \) bounds a negative-definite 4-manifold \( X \). Then there is a lattice embedding \( \phi : (\mathbb{Z}^n, Q_X) \to (\mathbb{Z}^r, -I) \).

**Construction**

One way to show that a link is \( \chi \)-slice is by searching for the right ribbon moves. These ribbon moves consist of adding a band with any number of twists in it and then adjusting the linking accordingly.

Any number of ribbon moves can be made until the resulting link is an unlink. Then in order for the link to be \( \chi \)-slice, we want \( \chi = 1 \) where

\[
\chi = \# \text{ of unknots} - \# \text{ of ribbon moves}.
\]

Looking for ribbon moves like this can be helpful in some cases, but becomes difficult to use in generality due the large number of possible bands.

**Obstruction**

We start with a pretzel link \( L = P(p, q, r, s) \), with \( \frac{p}{4} + \frac{q}{4} + \frac{r}{4} + \frac{s}{4} > 0 \).

1. Suppose that \( L \) bounds a surface \( F \subseteq B^4 \) with \( \chi(F) = 1 \).

2. Construct the double branched covers \( \Sigma_2(L) \) and \( \Sigma_2(F) \) of \( L \) and \( F \) (3- and 4-manifolds respectively).

3. Build a negative-definite 4-manifold \( X \) with boundary \( \Sigma_2(L) \).

4. Glue \( X \) to \( \Sigma_2(F) \) along their boundary \( \Sigma_2(L) \) to get a closed, negative-definite, oriented, 4-manifold \( \Sigma_2(F) \).

By Donaldson’s theorem, there must be an embedding \((\ast)\).

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**References**


**Findings**

Throughout, we suppose \( \frac{p}{4} + \frac{q}{4} + \frac{r}{4} + \frac{s}{4} > 0 \). The following cases arise:

- If \( p, q, r, \) and \( s \) are greater than 0, the only \( \chi \)-slice links are \( P(1, 1, 1, 1) \) and \( P(0, 1, 1, 1) \).

- In fact, an \( n \)-stranded pretzel link with all positive twists is \( \chi \)-slice if and only if it is of the form
  - (i) \( P(n + 1, 1, 1, 1) \)
  - (ii) \( P(n - 1, 1, 1, 1) \)

- If \( p, q, r > 0 \), \( s < 0 \), and \( L \) is \( \chi \)-slice, then either:
  - (i) Two of \( p, q, r \) are \( 2 \), the other one is any integer \( t \geq 1 \), and \( s \in \{ -1, -t, -(t + 4) \} \), or
  - (ii) One of \( p, q, r \) is \( 1 \), another is \( 3 \), the third is any integer \( t \geq 1 \), and \( s \in \{ -t, -(t + 3) \} \).

In the future, we would like to classify the remaining 4-stranded pretzel links and continue the general 4-stranded case. Here are some specific questions we still have:

1. Are \( P(2, t, 2, -t) \) and \( P(2, t, 2, -(t+4)) \) \( \chi \)-slice when \( t \equiv 2 \mod 4 \)?
2. For what values of \( p, q, r, \) and \( s \) is \( P(p, q, r, s) \) \( \chi \)-slice when two of these values are negative? This case has posed some complications because Donaldson’s theorem leaves several infinite families unobstructed. We have begun to use some other invariants to narrow down the list of potentially \( \chi \)-slice links.

**Applications of Donaldson’s Diagonalization Theorem**

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