

Numerical Approaches to Statistics on Ordered Set Partitions

Introduction

We study Lists of Lists (denoted $LLP_{n,k}$), which is the set of ordered set partitions of $\{1, 2, \ldots, n\}$ into k non-empty, disjoint subsets where the order of blocks and elements is considered. We study two particular statistics on $LLP_{n,k}$.

nsb

The statistic non-standard blocks (abbreviated as nsb) denotes the number of blocks that must be pushed to the right so that the smallest elements in each block are in increasing order.

nse

The statistic non-standard elements (abbreviated as nse) denotes the number of elements that must be pushed to the right so that the elements within each block are in increasing order.

Motivating Example

Let

$$\pi = 85 |2943| 716 \in LLP_{9,3}.$$

We arrange the blocks of π and elements within in increasing order: $\underline{85}|\underline{2943}|716 \Rightarrow \underline{58}|\underline{2349}|167 \Rightarrow 167|2349|58.$

The underlined blocks are non-standard blocks, and the blue elements are nonstandard elements, so $nsb(\pi) = 2$ and $nse(\pi) = 4$.

Generating Function

Herscovici [2] derived the following generating function that enumerates nsb and nse with respect to n and k:

$$S(n,k;u,v) = \sum_{\pi \in \mathsf{LLP}_{n,k}} u^{\mathsf{nsb}(\pi)} v^{\mathsf{nse}(\pi)} = \sum_{i=0}^{k} \sum_{j=0}^{n-k} \begin{bmatrix} n\\n-j \end{bmatrix} \begin{Bmatrix} n-j \\ k \end{Bmatrix} \begin{bmatrix} k\\k-i \end{bmatrix} u^{i} v^{j}.$$

Theorem on $\mathbb{E}[nsb]$

Define $\mathbb{E}[nsb_n]$ to be the expected value of nsb over all Lists of Sets of *n*-sets, and define $\mathbb{E}[nsb_{n,k}]$ to be the expected value of nsb over LLP_{n,k}.

$$\lim_{n \to \infty} \frac{\mathbb{E}[\mathsf{nsb}_n]}{n} = \frac{1}{\ln 4}$$

2. Fix k. Then $\mathbb{E}[\operatorname{nsb}_{n,k}] = k - H_k$.

Method of Approach

1. From the definition of expected value, we obtain that

$$\mathbb{E}[\mathsf{nsb}_n] = \frac{\sum_{k=1}^n k! (k - H_k) \left\{ \begin{array}{l} n \\ k \end{array} \right\}}{\sum_{k=1}^n k! \left\{ \begin{array}{l} n \\ k \end{array} \right\}}$$

Let

$$Q_n(t) = \sum_{k=1}^n k! (k - H_k) \left\{ {n \atop k} \right\} t^{k-1}; \quad P_n(t) = \sum_{k=1}^n k! \left\{ {n \atop k} \right\} t^{k-1};$$

Then $\mathbb{E}[\operatorname{nsb}_n] = \frac{Q_n(1)}{P_n(1)}$. Doron Zeilberger [Ekhad] proved that $\frac{P'_n(1)}{P_n(1)} \to \frac{n}{\ln 4}$.

https://math.gatech.edu/undergraduate-research

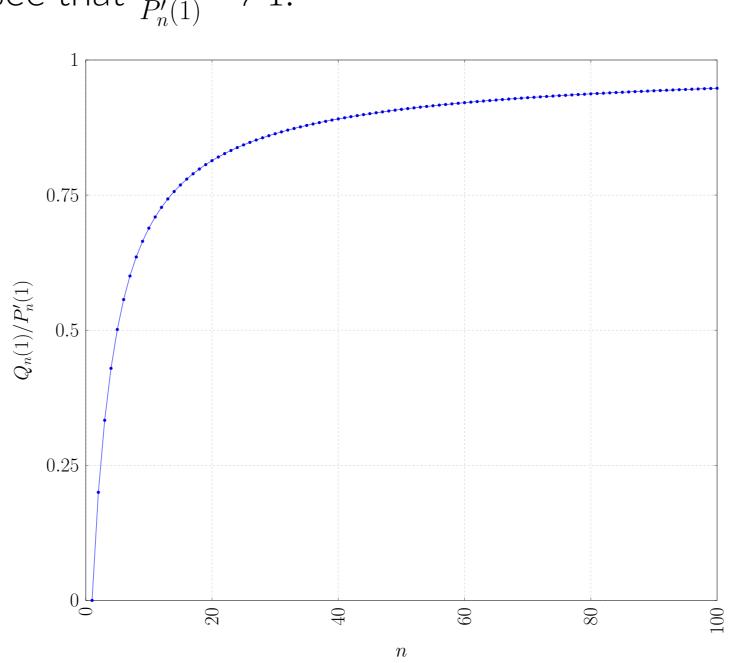
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Method of Approach (Continued)

1. Numerically, we see that $\frac{Q_n(1)}{P'(1)} \rightarrow 1$:





Thus,

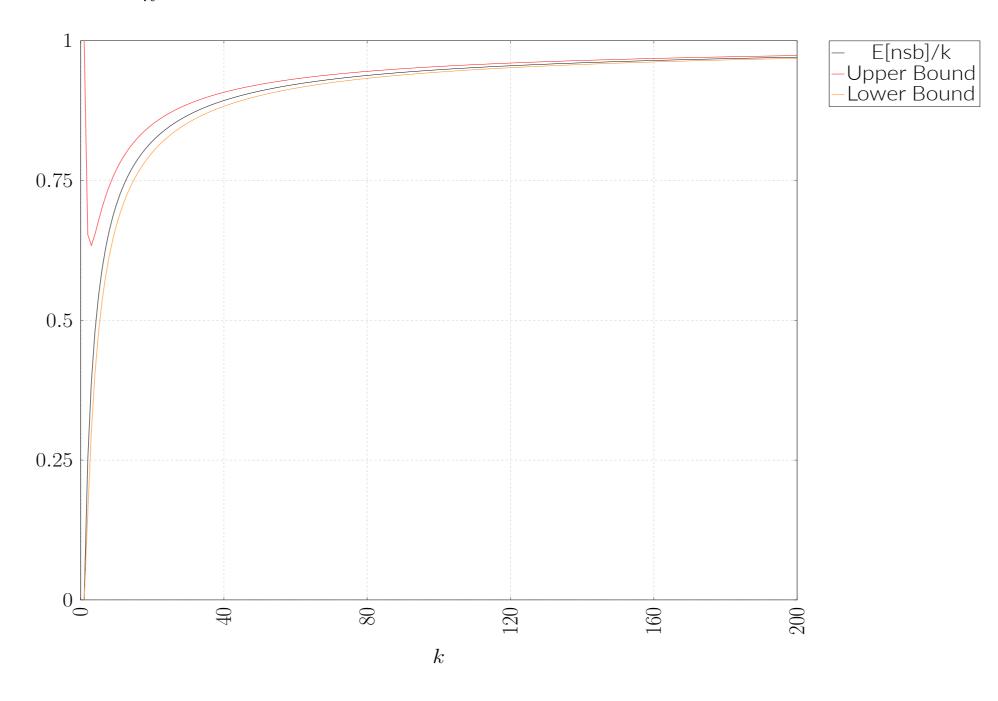
2. \

We can manipulate an explicit formula for $\mathbb{E}[nsb_{n,k}]$ using generating functions:

$$\mathbb{E}[\mathsf{nsb}_{n,k}] = \frac{1}{k!} \sum_{i=0}^{k-1} i = \frac{1}{k!} \left(1^{\overline{k}} - 1 \frac{d}{dx} [1^{\overline{k}}] \right) = k - H_k.$$

Bounding Approach

We can bound $\mathbb{E}[\operatorname{nsb}_{n,k}]$ above by $k - \ln k$ and below by $k - \ln k - 1$, which are the tightest bounds for H_k . The below graph lends support to show that $\mathbb{E}[\mathsf{nsb}_{n,k}] = k - H_k.$



Theorem on $\mathbb{E}[\mathbf{r}]$

Define $\mathbb{E}[nse_{n,k}]$ to be the expected value of n to be fixed for the following statements.

$$\lim_{n \to \infty} \frac{\mathbb{E}[\operatorname{nse}_{n,k}]}{n} =$$
$$\lim_{n \to \infty} \mathbb{E}[\operatorname{nse}_{n,n-d}] =$$
$$\max_{k \to \infty} \operatorname{var}[\operatorname{nse}_{n,n-d}] =$$

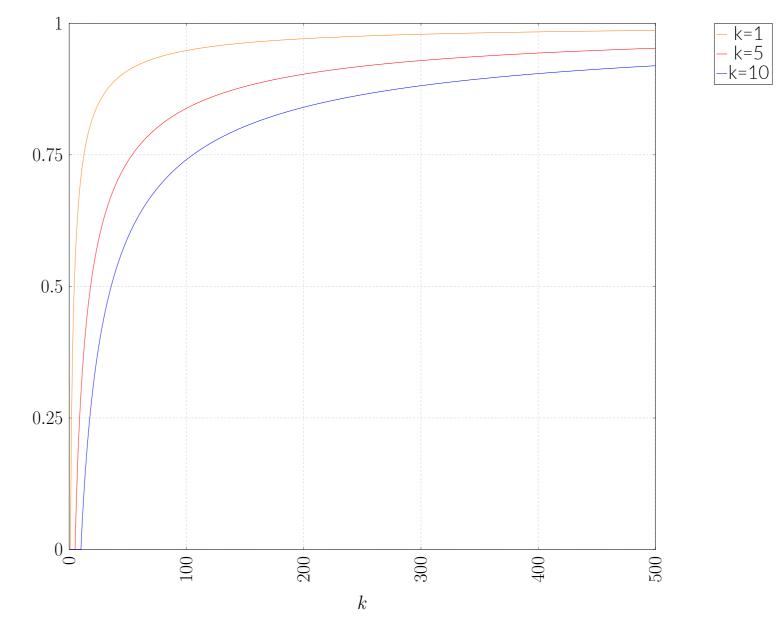
 $\left. \begin{array}{c} n \\ k \end{array} \right\} t^k.$

$$\lim_{n \to \infty} \frac{\mathbb{E}[\operatorname{nsb}_n]}{n} = \lim_{n \to \infty} \frac{Q_n(1)}{nP_n(1)} = \frac{1}{\ln 4}.$$

at the same result.

| nse] | |
|--------------------------------------|---------|
| ise over LLP $_{n,k}$. Assume k a | and d |
| = 1 | (1) |
| $=rac{d}{2} d$ | (2) |
| $=\frac{d}{4}$ | (3) |
| | |

although convergence is slower.



2. We can use two different approaches. Numerical Approach

| n | k=1 | k=2 | k=3 | k=4 | k=5 | k=6 | k=7 |
|---|----------|----------|----------|----------|----------|----------|-----|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0.5 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 1.166667 | 0.5 | 0 | 0 | 0 | 0 | 0 |
| 4 | 1.916667 | 1.111111 | 0.5 | 0 | 0 | 0 | 0 |
| 5 | 2.716667 | 1.791667 | 1.083333 | 0.5 | 0 | 0 | 0 |
| 6 | 3.55 | 2.52 | 1.725 | 1.066667 | 0.5 | 0 | 0 |
| 7 | 4.407143 | 3.283333 | 2.41 | 1.683333 | 1.055556 | 0.5 | 0 |
| 8 | 5.282143 | 4.073469 | 3.128571 | 2.339048 | 1.654762 | 1.047619 | 0.5 |

Consider the above table that displays $\mathbb{E}[nse_{n,k}]$. Moving along the diagonal of the table (where d is held constant), we see that $\mathbb{E}[\operatorname{nse}_{n,n-d}] \to \frac{d}{2}$.

Heuristic Combinatorial Approach

Define $\pi \in LLP_{n,n-d}$ to be *basic* if it only consists of singletons and pairs. By a combinatorial argument,

 $|\text{basic } \pi \in \text{LLP}_{n,n-d}| = \frac{(n-d)^{\underline{a}}}{\underline{a}}$ $\overline{|\mathsf{LLP}}_{n,n-d}|$ $(n-1)^{\underline{a}}$ As $n \to \infty$, most $\pi \in LLP_{n-d}$ are basic. Given some basic $\pi \in LLP_{n-d}$, $\mathbb{E}[\operatorname{nse}(\pi)] = \frac{d}{2}$, since for each of the d pairs we expect to move $\frac{1}{2}$ elements to the right. A bounding argument can be used to rigorously show that

 $\mathbb{E}[\operatorname{nse}_{n,n-d}] \to \frac{d}{2}.$

3. This result follows from a similar argument to (2).

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[2] Herscovici, O. (2022). Study of the p, q-deformed Touchard polynomials.



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Method of Approach

1. We can see that $\frac{\mathbb{E}[nse_{n,k}]}{n} \to 1$ numerically. These trends continue for larger k,

Acknowledgements

References

[Ekhad] Ekhad, Shalosh Zeilberger, D. The expected number of blocks in an ordered set partition of n objects is $n/\log(4) + o(1)$, its variance is (n/log(4))(1/log(4)1/2) + o(1), and it is asymptotically normal! (an experimental-

mathematical proof).