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## Introduction

We study Lists of Lists (denoted $\operatorname{LLP}_{n, k}$ ), which is the set of ordered set partitions of $\{1,2, \ldots, n\}$ into $k$ non-empty, disjoint subsets where the order of blocks and elements is considered. We study two particular statistics on LLP $n, k$.
nsb
The statistic non-standard blocks (abbreviated as nsb) denotes the number of blocks that must be pushed to the right so that the smallest elements in each block are in increasing order.

## nse

The statistic non-standard elements (abbreviated as nse) denotes the number of elements that must be pushed to the right so that the elements within each block are in increasing order.

## Motivating Example

Let

$$
\pi=85|2943| 716 \in \mathrm{LLP}_{9,3} .
$$

We arrange the blocks of $\pi$ and elements within in increasing order:

$$
\underline{85}|\underline{2943}| 716 \Rightarrow \underline{58}|\underline{2349}| 167 \Rightarrow 167|2349| 58 .
$$

The underlined blocks are non-standard blocks, and the blue elements are nonstandard elements, $\operatorname{sonsb}(\pi)=2$ and $\operatorname{nse}(\pi)=4$.

## Generating Function

Herscovici [2] derived the following generating function that enumerates nsb and nse with respect to $n$ and $k$ :

$$
S(n, k ; u, v)=\sum_{\pi \in \mathrm{LL} \mathrm{P}_{n, k}} u^{\mathrm{nsb}(\pi)} v^{\mathrm{nse}(\pi)}=\sum_{i=0}^{k} \sum_{j=0}^{n-k}\left[\begin{array}{c}
n \\
n-j
\end{array}\right]\left\{\begin{array}{c}
n-j \\
k
\end{array}\right\}\left[\begin{array}{c}
k \\
k-i
\end{array}\right] u^{i} v^{j} .
$$



## Method of Approach

1. From the definition of expected value, we obtain that

$$
\mathbb{E}\left[\text { nsb }_{n}\right]=\frac{\sum_{k=1}^{n} k!\left(k-H_{k}\right)\left\{\begin{array}{l}
n \\
k
\end{array}\right\}}{\sum_{k=1}^{n} k!\left\{\begin{array}{l}
n \\
k
\end{array}\right\}}
$$

Let

$$
Q_{n}(t)=\sum_{k=1}^{n} k!\left(k-H_{k}\right)\left\{\begin{array}{l}
n \\
k
\end{array}\right\} t^{k-1} ; \quad P_{n}(t)=\sum_{k=1}^{n} k!\left\{\begin{array}{l}
n \\
k
\end{array}\right\} t^{k} .
$$

Then $\mathbb{E}\left[\mathrm{nsb}_{n}\right]=\frac{Q_{n}(1)}{P_{n}(1)}$. Doron Zeilberger [Ekhad] proved that $\frac{P_{n}^{\prime}(1)}{P_{n}(1)} \rightarrow \frac{n}{\ln 4}$

## Method of Approach (Continued)

## Theorem on $\mathbb{E}[$ nse $]$

Define $\mathbb{E}\left[\right.$ nse $\left._{n, k}\right]$ to be the expected value of nse over $\operatorname{LLP}_{n, k}$. Assume $k$ and $d$ to be fixed for the following statements.
$\lim _{n \rightarrow \infty} \frac{\mathbb{E}\left[\text { nse }_{n, k}\right]}{n}=1$

1. Numerically, we see that $\frac{Q_{n}(1)}{P_{n}^{(1)}} \rightarrow 1$ :


Thus,

$$
\lim _{n \rightarrow \infty} \frac{\mathbb{E}\left[\mathrm{nsb}_{n}\right]}{n}=\lim _{n \rightarrow \infty} \frac{Q_{n}(1)}{n P_{n}(1)}=\frac{1}{\ln 4} .
$$

2. We can use two different methods to arrive at the same result.

## Analytical Approach

We can manipulate an explicit formula for $\mathbb{E}\left[\right.$ nsb $\left._{n, k}\right]$ using generating functions:

$$
\mathbb{E}\left[\mathrm{nsb}_{n, k}\right]=\frac{1}{k!} \sum_{i=0}^{k-1} i=\frac{1}{k!}\left(1^{\bar{k}}-1 \frac{d}{d x}\left[1^{\bar{k}}\right]\right)=k-H_{k} .
$$

## Bounding Approach

We can bound $\mathbb{E}\left[\right.$ nsb $\left._{n, k}\right]$ above by $k-\ln k$ and below by $k-\ln k-1$, which are the tightest bounds for $H_{k}$. The below graph lends support to show that $\mathbb{E}\left[\mathrm{nsb}_{n, k}\right]=k-H_{k}$


$$
\begin{align*}
n \rightarrow \infty & n  \tag{1}\\
\lim _{n \rightarrow \infty} \mathbb{E}\left[\text { nse }_{n, n-d}\right] & =\frac{d}{2}  \tag{2}\\
\lim \operatorname{var}\left[\mathrm{nse}_{n, n-d}\right] & =\frac{d}{2}
\end{align*}
$$

## Method of Approach

1. We can see that $\frac{\mathbb{E}\left[n s_{n}, k\right]}{n} \rightarrow 1$ numerically. These trends continue for larger $k$, although convergence is slower.

2. We can use two different approaches.

Numerical Approach

| $\mathbf{n}$ | $\mathrm{k}=1$ | $\mathrm{k}=2$ | $\mathrm{k}=3$ | $\mathrm{k}=4$ | $\mathrm{k}=5$ | $\mathrm{k}=6$ | $\mathrm{k}=\mathbf{7}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0.5 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 1.166667 | 0.5 | 0 | 0 | 0 | 0 | 0 |
| 4 | 1.916667 | 1.111111 | 0.5 | 0 | 0 | 0 | 0 |
| 5 | 2.716667 | 1.791667 | 1.083333 | 0.5 | 0 | 0 | 0 |
| 6 | 3.55 | 2.52 | 1.725 | 1.066667 | 0.5 | 0 | 0 |
| 7 | 4.407143 | 3.283333 | 2.41 | 1.683333 | 1.055556 | 0.5 | 0 |
| 8 | 5.282143 | 4.073469 | 3.128571 | 2.339048 | 1.654762 | 1.047619 | 0.5 |

Consider the above table that displays $\mathbb{E}\left[\right.$ nse $\left._{n, k}\right]$. Moving along the diagonal of the table (where $d$ is held constant), we see that $\mathbb{E}\left[\right.$ nse $\left._{n, n-d}\right] \rightarrow \frac{d}{2}$.

## Heuristic Combinatorial Approach

Define $\pi \in \operatorname{LLP}_{n, n-d}$ to be basic if it only consists of singletons and pairs. By a combinatorial argument,

$$
\frac{\mid \text { basic } \pi \in \operatorname{LLP}_{n, n-d} \mid}{\left|\mathrm{LLP}_{n, n-d}\right|}=\frac{(n-d)^{\underline{d}}}{(n-1)^{\underline{d}}} \text {. }
$$

As $n \rightarrow \infty$, most $\pi \in \operatorname{LLP}_{n-d}$ are basic. Given some basic $\pi \in L L P_{n-d}$, $\mathbb{E}[$ nse $(\pi)]=\frac{d}{2}$, since for each of the $d$ pairs we expect to move $\frac{1}{2}$ elements to the right. A bounding argument can be used to rigorously show that $\mathbb{E}\left[\mathrm{nse}_{n, n-d}\right] \rightarrow \frac{d}{2}$
This result follows from a similar argument to (2).

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## References

[Ekhad] Ekhad, Shalosh Zeilberger, D. The expected number of blocks in an ordered set partition of $n$ objects is $n / \log (4)+o(1)$, it variance is $(n / \log (4))(1 / \log (4) 1 / 2)+o(1)$, and it is asymptotically norma!! (an experimentalmathematical proof

