Generalizing Quasi-Triangular Hopf Algebras and Braided Monoidal Categories

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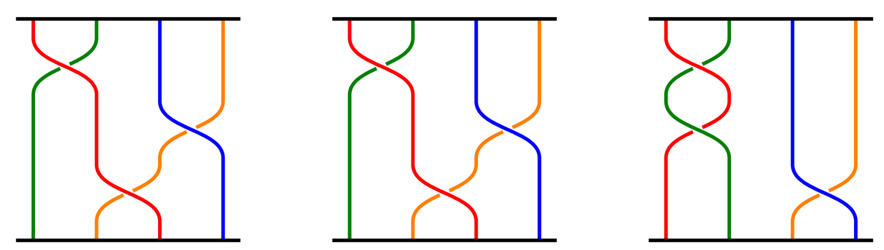
Abstract

We introduce singularly braided monoidal categories, a generalization of braided monoidal categories, and extend the analogous coherence results of Joyal and Street. Braided monoidal categories arise in knot theory since they have a natural action by the braid group. Analogously, our singularly braided monoidal categories are acted on by the singular braid monoids. Many knot invariants arise via actions of the braid group on the braided monoidal category of representations of a hopf algebra. These categories are especially simple to work with since a braided structure on $\mathrm{Rep}(H)$ is equivalent to a choice of a single element in $H \otimes H$ satisfying a small list of axioms. We also generalize quasi-triangular hopf algebras to a new structure whose representations are acted on by the singular braid monoids, but their category of representations is *not* singularly braided monoidal.

Background: Braids and Monoidal Categories

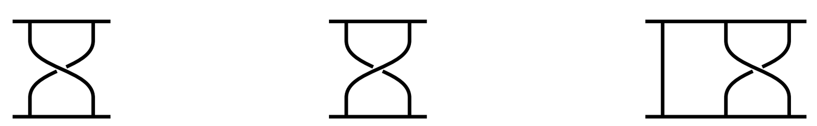
A **braid** is a collection of n strands between n points. Below we have various braids on four strands.



The braids on n strands form the **braid group** B_n whose group product is braid composition.

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}^{(1)} \quad \sigma_i\sigma_j = \sigma_j\sigma_i^{(2)} \right\rangle.$$

Relation (1) is known as the **braid relation**, and (2) holds for $|i-j| > 1$. Examples of generators are:



σ_1 σ_1^{-1} σ_2

A **monoidal category** M is a "categorification" of a monoid, consisting of:

- a collection of objects X, Y, Z...
- a collection of arrows between objects
- a multiplication $\otimes : M \times M \to M$
- a unit object 1 of M

Examples include the monoidal categories **Set** and **Vect**.

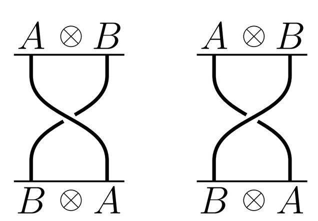
	Set	**Vect**
Objects	All sets X	All vector spaces V
Arrows	Functions $f : X \to Y$	Linear transformations $T : U \to V$
Multiplication	Cartesian product $\times$	Tensor Product $\otimes$

A **braided monoidal category** M is a monoidal category with a notion of an invertible braiding, or a twist arrow

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

for each pair of objects X, Y. Such a category allows us to envision the twist morphisms as braids. Below we have $c_{A,B}$ and $c_{A,B}^{-1}$ interpreted geometrically.

Braided monoidal categories satisfy a few axioms which make them a good model for the geometry of the braid group. More precisely, the braid groups act on a braided monoidal category.



geometry $\xrightarrow{\text{braided monoidal categories}}$ Category Theory

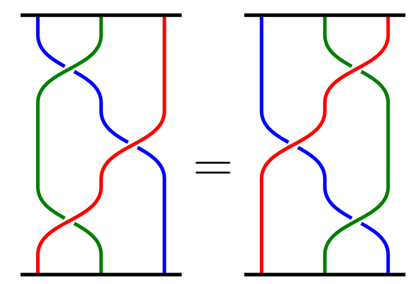
Understanding Braids Algebraically

A representation of the braid group B_n is a vector space V along with a group homomorphism

$$\rho : B_n \to \mathrm{GL}(V)$$

geometry $\xrightarrow{\text{representation}}$ algebra

Such a representation allows us to pass from geometry to algebra. Specifically, if we have representations $\{B_n \to \mathrm{GL}(V^{\otimes n})\}$ for each n, we can turn braids into linear operators, or matrices, and can sometimes even use this to construct **knot invariants**. To define a representation we just need to specify what happens at a crossing, i.e. give an invertible linear transformation R of $V \otimes V$ satisfying



$$(R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) = (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R)$$

This is known as the **Yang-Baxter equation**, and invertible solutions are called ***R*-matrices**. The Yang-Baxter equation is the algebraic analogue of the braid relation.

Quasi-triangular Hopf Algebras

A quasi-triangular Hopf algebra is a hopf algebra with an invertible element $R \in H \otimes H$ called the **universal *R*-matrix**, satisfying certain conditions. Write $R = \sum_i a_i \otimes b_i$, and define $R_{12} = \sum_i a_i \otimes b_i \otimes 1$, $R_{13} = \sum_i a_i \otimes 1 \otimes b_i$, and $R_{23} = \sum_i 1 \otimes a_i \otimes b_i$. Then R is a universal R-matrix if

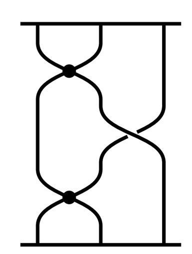
$$\begin{gathered} R \cdot \Delta(a) = \tau(\Delta(a)) \cdot R \\ (\Delta \otimes \mathrm{id}_H)(R) = R_{13}R_{23} \\ (\mathrm{id}_H \otimes \Delta)(R) = R_{13}R_{12} \end{gathered}$$

where $\tau(x \otimes y) = y \otimes x$.

Theorem. If H is a quasi-triangular Hopf algebra with universal R-matrix R, then R satisfies the "quantum" Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. For any representation V of H, multiplication by R on $V \otimes V$ satisfies the Yang-Baxter equation, and so is an R-matrix.

Singularly Braided Monoidal Categories

The **singular braid monoid** SB_n is the monoid containing the braid group B_n with the additional singular crossings $\tau_1, \ldots, \tau_{n-1}$ where τ_i represents a **singular crossing** of two strands. Such a singular braid is pictured on the right. These satisfy the relation $\sigma_i\tau_j = \tau_j\sigma_i$ and the mixed braid relations



$$\begin{gathered} \sigma_i\sigma_{i+1}\tau_i = \tau_{i+1}\sigma_i\sigma_{i+1} \\ \tau_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\tau_{i+1} \end{gathered}$$

Generalizing braided monoidal categories, we define a **singularly braided monoidal category** to be a braided monoidal category with an additional family of twist morphisms $s_{X,Y} : X \otimes Y \to Y \otimes X$ for all X, Y which are not necessarily invertible. Like how the braid groups act on any braided monoidal category, the singular braid monoids act on any singular braided monoidal category.

Q-matrices and (R, Q)-Hopf Algebras

Let V be a free module of finite rank with an R-matrix $R \in \mathrm{GL}(V \otimes V)$. A linear endomorphism $Q : V \otimes V \to V \otimes V$ satisfying the **mixed Yang-Baxter equations**

$$\begin{gathered} RQ = QR \\ (Q \otimes id) \circ (id \otimes R) \circ (R \otimes id) = (id \otimes R) \circ (R \otimes id) \circ (id \otimes Q) \\ (R \otimes id) \circ (id \otimes R) \circ (Q \otimes id) = (id \otimes Q) \circ (R \otimes id) \circ (id \otimes R) \end{gathered}$$

which is the algebraic analogue of the singular braid relations, is called a Q-**matrix** associated to R. Given this data, we get a family of singular braid monoid representations $\{SB_n \to \mathrm{End}(V^{\otimes n})\}$ where an overcrossing becomes R and a singular crossing Q.

An (R, Q)-**Hopf algebra** is a quasi-triangular hopf algebra (H, R) with a (possibly not invertible) element $Q \in H \otimes H$ satisfying $RQ = QR$ and $R\Delta(a) = \tau(\Delta(a))R$.

Theorem. If H is an (R, Q)-Hopf algebra, then R and Q satisfy the "quantum" mixed Yang-Baxter equations $Q_{12}R_{13}R_{23} = R_{23}R_{13}Q_{12}$ and $R_{12}R_{13}Q_{23} = Q_{23}R_{13}R_{12}$. For any representation V of H, multiplication by R and Q on $V \otimes V$ satisfy the mixed Yang-Baxter equation, and so give an R-matrix and an assosciated Q-matrix.

Applications: Sweedler Hopf Algebra

The **Sweedler Hopf algebra** is a four dimensional unital associative algebra over C generated by two elements x, y subject to the relations

$$x^2 = 1 \quad y^2 = 0 \quad xy = -yx$$

It's a bialgebra with comultiplication $\Delta(x) = x \otimes x$ and $\Delta(y) = y \otimes 1 + 1 \otimes y$ and counit $\varepsilon(x) = 1$, $\varepsilon(y) = 0$. Furthermore it's a quasi-triangular hopf algebra with universal R-matrix

$$R = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + x \otimes 1 - x \otimes x) + y \otimes y + y \otimes z - z \otimes y + z \otimes z$$

We found that this hopf algebra admits a Q-matrix, and that all possible Q-matrices are of the form

$$Q = a(1 \otimes 1 + 1 \otimes x + x \otimes 1 - x \otimes x) + by \otimes y + c(y \otimes z - z \otimes y) + (4a - b)z \otimes z$$

for scalars a, b, c. Interestingly Q is invertible as long as $a \neq 0$ and it satisfies $(\Delta \otimes \mathrm{id}_H)(Q) = Q_{13}Q_{23}$ and $(\mathrm{id}_H \otimes \Delta)(Q) = Q_{13}Q_{12}$, even though we only required $QR = RQ$ and $Q\Delta(a) = \tau(\Delta(a))Q$.

References and Acknowledgments

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