Meanders

Ida De Vierno & Isabella Nang

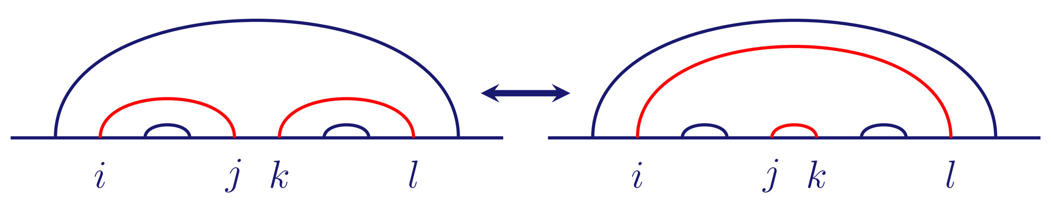
Georgia Tech IMPACT Math-Bio REU

Introduction

Meanders have been studied since 1912 [3], and it is still unknown how to enumerate them; however we know it grows exponentially. In our research we prove some properties of meanders and prove that at least a subset of the meander graph is connected. This subset also grows exponentially.

Definitions

1. A *meander* is a closed curve crossing a horizontal line $2n$ times that does not intersect itself.
2. A *non-crossing perfect matching* (NCPM) of order n is a set of n arcs. Each arc corresponds to a unique pair (x, y) with $x, y \in \{1, 2, \ldots, 2n\}$, $x < y$. Given a horizontal line with $2n$ points labeled in increasing order left to right, the arcs can be drawn above the line, connecting the points from their pair such that no two arcs intersection and every point is the endpoint of an arc.
3. Each NCPM corresponds to a sequence of As and Bs with A at the beginning of each arc and B at the ending. Note that this is a ballot sequence.
4. A *parent-child* is a pair of arcs (i, ℓ) and (j, k) with no arc (m, n) where $i < m < j$ and $k < n < l$.
5. Two arcs (i, j) and (k, ℓ) are *siblings* if $k = j + 1$.
6. A *local move on a NCPM* replaces siblings (i, j) and (k, ℓ) with parent child arcs (i, ℓ) and (j, k) or vice versa.

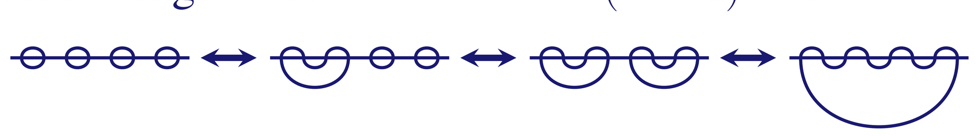


7. A *local move on a meander* consists of a local move on the NCPM above the horizontal line and an independent local move on the NCPM below the line, provided the result is a meander.
8. The *state space graph of NCPMs* is the graph whose vertices are NCPMs of order n and there is an edge between two NCPMs if one can be transformed into the other via a single local move. The state space graph of meanders is defined similarly.

Non-Crossing Perfect Matchings
State Space Graph

Distance between NCPMs to Form a Meander is $\geq n - 1$

Draw the same NCPM on the top and bottom of a horizontal forming n curves. Now do moves that only merge curves strictly on the bottom until a meander is formed. We will need to do at least $n - 1$ local moves to form a single curve because $n - (n - 1) = 1$.



s-Value

For an arc (x, y) define its s-value as $s(x, y) = \frac{y-x-1}{2}$.
Given a NCPM $\beta = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ define

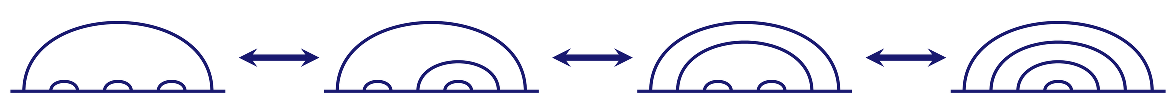
$$s_\beta = \sum_{i=1}^{n} s(x, y).$$

Define an even NCPM as a NCPM with an even s-value and an odd NCPM as a NCPM with an odd s-value.

The state space graph of NCPMs is bipartite and connected.

Bipartite: Group NCPMs with even s-values into one class and those with odd s-values into another. We will show that all edges run between these classes. For some arbitrary NCPM β, make a local move.
Suppose the move is a sibling move on the arcs (x, y) and $(y + 1, z)$. When the local move occurs, β' results with (x, y) and $(y + 1, z)$ replaced by (x, z) and $(y, y + 1)$.
$s'_\beta = s_\beta - s(x, y) - s(y+1, z) + s(x, z) + s(y, y+1) = s_\beta + 1$.
If the move was a parent-child move, then $s'_\beta = s_\beta - 1$.

Connectedness: Given a NCPM, make sibling to parent-child moves, increasing the s-value. Continue making these moves until there are no longer any siblings, resulting in the rainbow which has the maximum s-value.



Rainbow

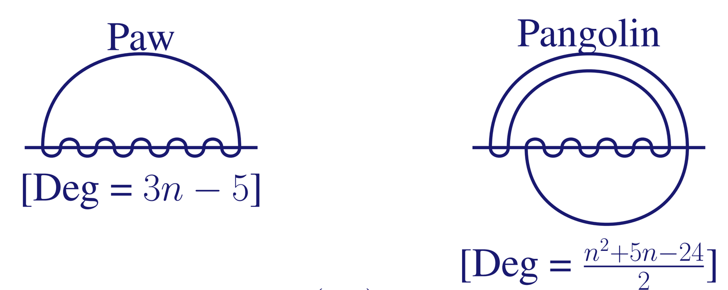
State Space Graph of Meanders

The state space graph of meanders is bipartite.

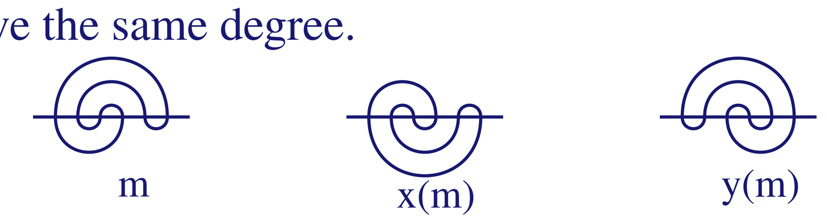
For odd order n: a meander is composed of two odd or two even NCPMs. A local move changes the parities of both s-values of the NCPMs. Hence, $\frac{odd}{odd} \leftrightarrow \frac{even}{even}$.
For even order n: a meander is composed of an even and odd NCPM. Similarly, $\frac{odd}{even} \leftrightarrow \frac{even}{odd}$.

Degree of Meanders

The *degree* of a meander is the number of neighbors it has in the meander state space graph.
Below are two important meanders and their degrees.

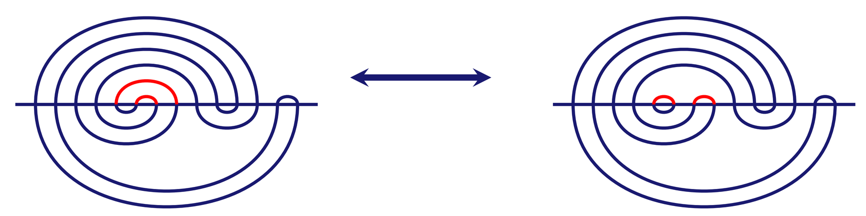


For a meander m, let $x(m)$ be the flip over a horizontal line of m and $y(m)$ a flip across a vertical line. If a meander n is a neighbor of m, then $x(n)$ is a neighbor of $x(m)$ and $y(n)$ is a neighbor of $y(m)$. All symmetries of m have the same degree.

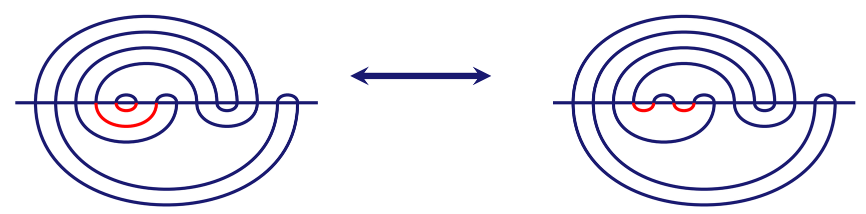


Local Moves on Meanders

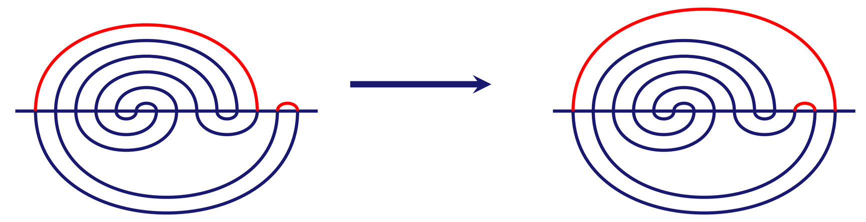
Given a meander, any local move on the top will result in two closed curves



A local move affecting two arcs from separate curves will merge them



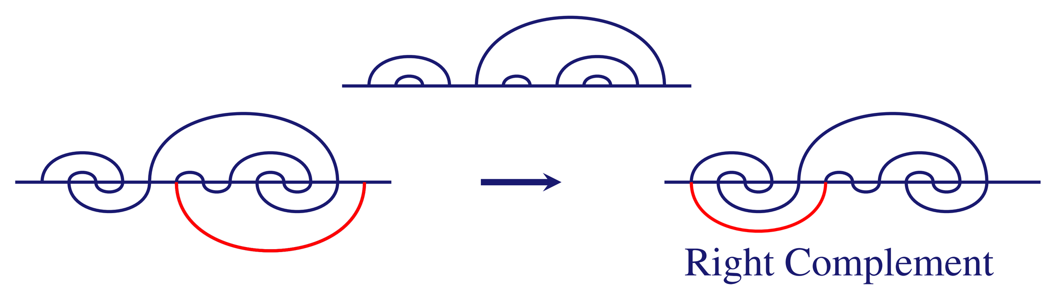
Not all local moves done on top has a corresponding move on the bottom to merge two curves



The difficulty of proving connectedness of the meander graph is evident in the above example.

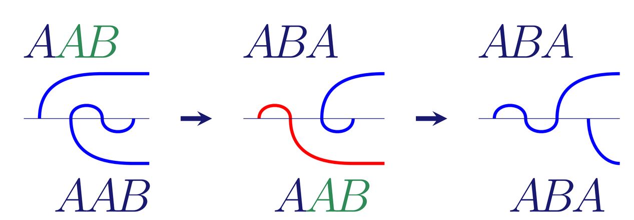
Connectivity of Kreweras Complements

Draw the same NCPM on the bottom shifted one position to the right. Replace arc $(x, 2n + 1)$ with $(1, x)$.

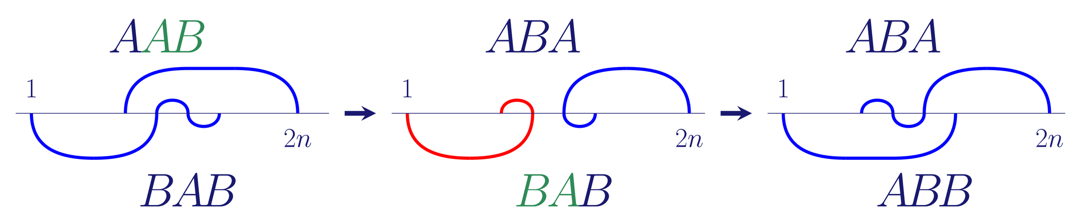


The total number of complements is $2\frac{1}{n+1}\binom{2n}{n} - 2$

There are $\frac{1}{n+1}\binom{2n}{n}$ right complement meanders. A similar calculation for the left complements can be done (including a correction for the paw).
Any complement meander can be transformed into the paw which is also a complement using two types of moves.
Type 1:

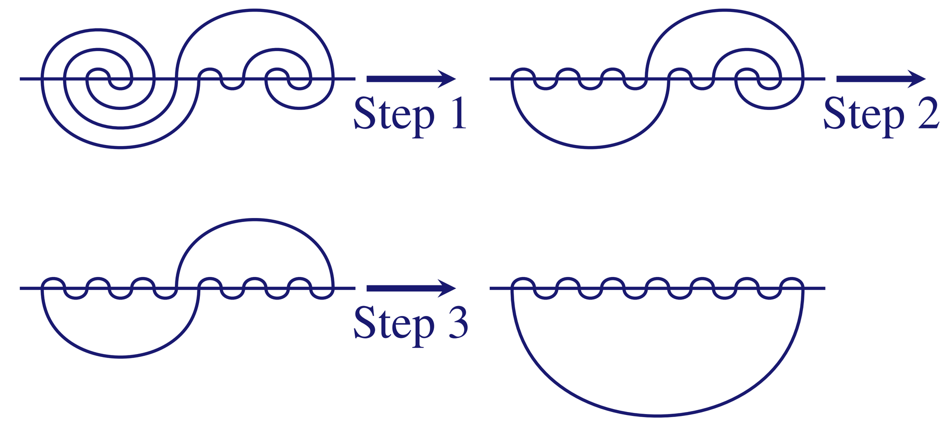


Type 2:



Algorithm to Tranform Complements into Paw

1. Make the left most type 1 move before the top arch $(i, 2n)$. Repeat until no more type 1 moves are available in the range.
2. Make the left most type 1 move under the top arch $(i, 2n)$. Repeat until no more type 1 moves are available in the range.
3. Then do the appropriate number of type 2 moves to get $[1, 2n]$ as ABAB...AB.



Note: After each type of move the resulting meander remains a complement.
The left complement is shifted one position to the left. Replace arc $(0, x)$ with $(x, 2n)$.
y(left complement) = right complement. The same algorithm can be applied to obtain the paw.

Future Research

- Show the meander graph is connected.
- Define a Markov Chain by our local moves on meanders.
- Determine if it is rapidly mixing and use it to estimate the number of meanders.

References

[1] C. E. Heitsch and P. Tetali. Meander graphs. *DMTCS Proceedings*, 2011.

[2] G. Kreweras. Sur les partitions non croisees d'un cycle. *Discrete Mathematics*, 1(4):333 – 350, 1972.

[3] H. Poincaré. Sur un théorème de géométrie. *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, 33(1):375–407, 1912.

Acknowledgements

We would like to thank Tom Prag, Fidel Barrera-Cruz, Heather Smith, Christine Heitsch for their contribution and input in our research. This research was supported by the NSF DMS grant #1344199.

Contact Information

inang1@student.gsu.edu
ivierno3@gatech.edu