Invariant Measures of an Arnold Cat Map.

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End of REU Conference - 7/24/2020

Plan of the talk

- Introduction to the Arnold cat map as a chaotic dynamical system.
- The calculation of an invariant manifold using a local perturbation.
- Meaning of an invariant measure.
- How big does it get?

Consider any point ψ in the unit square. Now apply the mapping $S_0\psi$, where

$$S_0 \equiv \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

This would then look like

$$S(\psi) = \begin{pmatrix} \psi_1 + \psi_2 \\ \psi_1 \end{pmatrix}.$$

Consider this map's indefinite iteration. Clearly this would leave the unit square domain quickly, so we define it on the 2-torus

$$\mathbb{T}^2 \equiv \mathbb{R}^2 / 2\pi\mathbb{Z}^2,$$

which would give us

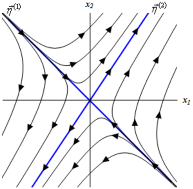
$$S(\psi) = \begin{pmatrix} \psi_1 + \psi_2 \\ \psi_1 \end{pmatrix} \mod 2\pi.$$

This map is well-defined, and similar maps are commonly referred to as Arnolds cat map, named after the first illustrative example generated by Arnold. In his study, he applied the mapping to an image of a cat to demonstrate chaotic dynamics.

Consider iterations as time, such that the Nth iteration serves as the state of the system at $t = N$. Eigenvector analysis of the transformation demonstrates the hyperbolicity of the system. We call $\mathbf{v}_+$, $\mathbf{v}_-$ the two normalized eigenvectors relative to the eigenvalues $\frac{1 \pm \sqrt{5}}{2}$. We also express λ_+ and λ_- in terms of $\lambda = \frac{\sqrt{5}-1}{2}$, the inverse of the maximum eigenvalue of S_0 as follows,

$$\lambda_+ = \frac{1}{\lambda} \quad , \quad \lambda_- = -\lambda. \tag{1}$$

Hyperbolic dynamical systems are characterized by their possession of both a stable and unstable dimension in phase space. Due to a useful characteristic of hyperbolic systems, this analysis can be transferred to the study of local behavior of nonlinear systems where the stable and unstable subspaces are replaced by local stable and unstable manifolds.



Now, what happens when we introduce a local perturbation? We can say

$$S_\epsilon(\psi) = \left(\begin{pmatrix} \psi_1 + \psi_2 \\ \psi_1 \end{pmatrix} + \epsilon \begin{pmatrix} f_1(\psi_1, \psi_2) \\ f_2(\psi_1, \psi_2) \end{pmatrix} \right) \mod 2\pi$$

so that we can study the perturbation

$$S_\epsilon \varphi = S_0 \varphi - \epsilon \mathbf{f}(\varphi).$$

When we perturb a system, we take a system in dynamic equilibrium and push it a bit in some arbitrary direction. We then analyze what effects this small variation of order ϵ has. Doing so, and utilizing Taylor's Theorem to form series representations of each function gives rise to the work I have completed this summer.

It is important to note that **f** is a real-valued trigonometric polynomial given by

$$\mathbf{f}(\varphi) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^2} e^{i\boldsymbol{\nu} \cdot \varphi} \mathbf{f}_{\boldsymbol{\nu}}$$

such that $|\boldsymbol{\nu}| \leq N$.

To perform this, we again write out the expansion of $\mathbf{h}(\psi)$, followed by $S_0\mathbf{h}(\psi)$, $\mathbf{h}(S_0\psi)$, and $\epsilon\mathbf{f}(\psi + h(\psi))$. We have

$$\mathbf{h}(\psi) = \epsilon\mathbf{h}^{(1)}(\psi) + \epsilon^2\mathbf{h}^{(2)}(\psi) + \cdots + \epsilon^i\mathbf{h}^{(i)}(\psi) + \cdots ,$$

$$S_0\mathbf{h}(\psi) = \epsilon S_0\mathbf{h}^{(1)}(\psi) + \epsilon^2 S_0\mathbf{h}^{(2)}(\psi) + \cdots + \epsilon^i S_0\mathbf{h}^{(i)}(\psi) + \cdots ,$$

and

$$\mathbf{h}(S_0\psi) = \epsilon\mathbf{h}^{(1)}(S_0\psi) + \epsilon^2\mathbf{h}^{(2)}(S_0\psi) + \cdots + \epsilon^i\mathbf{h}^{(i)}(S_0\psi) + \cdots .$$

Now it is time to construct a power series expansion for $\mathbf{f}$. Doing so, we find that

$$\mathbf{f}(\psi + \mathbf{h}(\psi)) = \mathbf{f}(\psi) + \epsilon \begin{pmatrix} \partial_{\psi_1} f_1(\psi) & \partial_{\psi_2} f_1(\psi) \\ \partial_{\psi_1} f_2(\psi) & \partial_{\psi_2} f_2(\psi) \end{pmatrix} \mathbf{h}(\psi) + \cdots + \frac{\epsilon^i}{i!}\partial_\psi^{(i-1)}\mathbf{f}\,\mathbf{h}(\psi) + \cdots .$$

It is important to notice the vertical column of arbitrary terms formed by these expansions. If we take every term with an ϵ out front, it is clear where each order's expression comes from.

Why Periodic Boundary Conditions?

It turns out that the modulus operator is the entity actually manifesting the chaotic nature of this system. The 'wrapping around' as the system evolves generates the ferocious mixing. For an unperturbed Arnold cat map, (nearly) any initial trajectory densely fills the phase space. This divergence from near trajectories in time is described by the lyupanov exponents of the system, and is the defining factor of the dynamical chaos.

Let us begin by considering the existence of a function $H : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ such that $H \circ S_0 = S_\epsilon \circ H$. We write

$$H(\psi) = \psi + \mathbf{h}(\psi).$$

Rearranging our definition of H shows us that $S_\epsilon = H \circ S_0 \circ H^{-1}$, so we see

$$S_\epsilon^n = H \circ S_0^n \circ H^{-1}$$

Because of the local product structure
We want an exact expression for the attractor that this map collapses onto, in terms of only the initial functions and characteristic parameters of the map. If we look closely at the perturbation and rearrange it a little bit, our first order of epsilon relation looks like this:

$$S_0 \mathbf{h}^{(1)}(\psi) - \mathbf{h}^{(1)}(S_0\psi) = \mathbf{f}(\psi) \tag{2}$$

Recall that it is convenient (and equally valid) when manipulating this local perturbation to work in the basis of the eigenvectors of the transformation S_0. This brings us to

$$\lambda_+ h_+^{(1)}(\psi) - h_+^{(1)}(S_0\psi) = f_+(\psi), \tag{3}$$

$$\lambda_- h_-^{(1)}(\psi) - h_-^{(1)}(S_0\psi) = f_-(\psi). \tag{4}$$

Analytic Solutions at the 1st Order

To solve equation (2), we must get creative. We replace the RHS of (2) with a general function, as follows. Let

$$\lambda_+ h_+^{(k)}(\psi) - h_+^{(k)}(S_0\psi) = \Omega(\psi), \tag{5}$$

or

$$h_+^{(k)}(\psi) = \lambda_+^{-1} h_+^{(k)}(S_0\psi) + \lambda_+^{-1}\Omega(\psi),$$

where $k \in \mathbb{N}$, and Ω is a function of $k-1$ orders of h_+, f_+, and its derivatives. It is important to note here that because this transformation is an automorphism, we can apply $S_0\psi \longrightarrow \psi$ to (2) to make another progressive step towards the analytic solution.

After repeating this process multiple times, it is possible to write a series representation of the solution based on generating each consecutive power of S_0, giving us

$$h_+^{(k)}(\psi) = \sum_{n=0}^{\infty} \lambda_+^{-(n+1)}\Omega(S_0^n\psi). \tag{6}$$

In this case, our expression becomes

$$h_+^{(1)}(\psi) = \sum_{n=0}^{\infty} \lambda_+^{-(n+1)} f_+(S_0^n\psi). \tag{7}$$

The solution to (3) is strikingly similar, the only differences are the powers of lambda and S_0 inside the function. We now must assure ourselves that this series converges.

Convergence

As we assess convergence of this series, we call the RHS of (7) Γ. We note that since **f** is a trigonometric polynomial, for some arbitrary argument φ, we know that

$$|\Gamma(\varphi)| \leq F_+,$$

where

$$F_+ \equiv \|f_+\|_\infty = \sup \Gamma,$$

Thus we can see that the sum $h_+^{(1)}$ must be bounded by the product of F_+ and $\left(\frac{1}{1-\lambda_+}\right)$, implying convergence.

This same line of thinking can hopefully reassure the audience that every order converges, as each order is only written in terms of previous orders of **h**, a prefactor of λ_+, and **f** and its derivatives (which are also bounded).

For the second order (after changing the basis), we get the $h_+^{(2)}$ relation to be

$$\lambda_+ h_+^{(2)}(\psi) - h_+^{(2)}(S_0\psi) = \partial_+ f_+(\psi) h_+^{(1)}(\psi) + \partial_- f_+(\psi) h_-^{(1)}(\psi), \tag{8}$$

a tad bit more daunting than the first order. Not to worry, we already have the solution with (5)! This gives us

$$h_+^{(2)}(\psi) = \sum_{n,m=0}^{\infty} \lambda_+^{-(n+m+2)} \partial_+ f_+(S_0^n\psi) f_+(S_0^{(m+n)}\psi)) - \sum_{n,m=0}^{\infty} \lambda_+^{-(1+n+m)} \partial_- f_+(S_0^n\psi) f_-(S_0^{(n-m-1)}\psi)),$$

Now let us have a look at the 3rd order...

After a nontrivial exercise in algebra, the final expression for $h_+^{(3)}$ is given by

$$
\begin{aligned}
h_+^{(3)}(\psi) = & \sum_{n,m,k=0}^{\infty} \lambda_+^{-(n+m+k+3)}\, \partial_+ f_+(S_0^k\psi)\, \partial_+ f_+(S_0^{(n+k)}\psi)\, f_+(S_0^{(n+m+k)}\psi) \\
& + \sum_{n,m,k=0}^{\infty} \lambda_+^{-(n+m-k+2)}\, \partial_+ f_+(S_0^k\psi)\, \partial_- f_+(S_0^{(n+k)}\psi)\, f_-(S_0^{(n+m+k-1)}\psi) \\
& + \sum_{n,m,k=0}^{\infty} \lambda_+^{-(n+m+k+2)}\, \partial_- f_+(S_0^k\psi)\, \partial_+ f_-(S_0^{(n+k)}\psi) f_+(S_0^{(m+k)}\psi) \\
& + \sum_{n,m,k=0}^{\infty} \lambda_+^{-(n+m+k+1)}\, \partial_- f_+(S_0^k\psi)\, \partial_- f_-(S_0^{-(n-k+1)}\psi) f_-(S_0^{-(m-k+1)}\psi) \\
& + \frac{1}{2}\sum_{n,k=0}^{\infty} \left(\lambda_+^{-(n+k+2)} f_+(S_0^{(n+k)}\psi)\right)^2 \partial_+^2 f_+(S_0^k\psi) \\
& - \sum_{n,k=0}^{\infty} \lambda_+^{-(2n+k+2)} f_+(S_0^{(n+k)}\psi)\, f_-(S_0^{-(n-k+1)}\psi) \partial_+\partial_- f_+(S_0^k\psi) \\
& + \frac{1}{2}\sum_{n,k=0}^{\infty} \left(\lambda_+^{-(n+k+1)} f_-(S_0^{-(n-k+1)}\psi)\right)^2 \partial_-^2 f_+(S_0^k\psi),
\end{aligned}
$$

We have analytically determined what we could easily numerically see, but seemed very out of reach initially. Better yet, we are able to determine exactly the invariant manifold that the system collapses onto as ϵ grows, in terms of no more than the starting information. We have the ability to use the lovely framework of measure theory to generalize our study of the chaotic dynamics.

If we think about this unit square as a probability space rather than a physical space, we can reach powerful new information about the system. The transformation is in fact measureable, so we define a measure μ on the measure space $(\mathbb{T}^2, \mathcal{B}, \boldsymbol{\lambda})$.

An invariant measure is defined to be a measure μ on $(\mathbb{T}^2, \mathcal{B}, \boldsymbol{\lambda})$ such that for a measureable function $\mathbf{f} : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ and any set $A \subseteq \mathbb{T}^2$

$$\mu\left(\mathbf{f}^{-1}(A)\right) = \mu(A)$$

Because this is a volume-preserving anosov diffeomorphism (even under small perturbation), many invariant measures exist!

Conclusions

The main takeaways from my work this summer are the following:

- The notion of a Arnold cat map as a dynamical system
- The use of perturbation theory to help with intense computation (such as determining the attractor manifold)
- The convergence of my analytic solution and its generality
- The prevelance of invariant measures in dynamical systems, and the importance of my future constructive study of coupled Arnold cat maps

The End

Thank You