Background

Latin Squares and Subsquares

An order-*n* Latin square is an $n \times n$ array of *n* symbols, such that each row and column contains each symbol exactly once.

An order-m subsquare of an order-n Latin square is the $m \times m$ array induced by some set of (not necessarily adjacent) *m* rows and *m* columns.

Let L be an order-n Latin square chosen uniformly at random.

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

order-4 Latin square with order-2 Latin subsquare

Conjecture (McKay and Wanless, 1999)

- As $n \to \infty$, the expected number of order-3 Latin subsquares of L tends to $\frac{1}{18}$.
- As $n \to \infty$, the probability L contains a Latin subsquare of order greater than 3 tends to zero.

Probabilistic Heuristic

For tuples (i, j, k) and (i', j', k'),

 $\mathbb{P}(\mathbf{L}_{\mathbf{ij}} = k \text{ and } \mathbf{L}_{\mathbf{i'j'}} = k') \approx \mathbb{P}(\mathbf{L}_{\mathbf{ij}} = k) \mathbb{P}(\mathbf{L}_{\mathbf{i'j'}} = k') = \frac{1}{n^2}$

Essentially, for large enough n and a small enough set of entries, we can approximate these events as independent.

Example: There are $\binom{n}{3}^3$ ways to choose the rows, columns, and symbols for an order-3 subsquare and $3! \times 2$ ways for it to be Latin. Hence, the expected number of order-3 Latin subsquares in L tends to

$$\lim_{n \to \infty} \frac{\binom{n}{3}^3 \cdot 3! \cdot 2}{n^9} = \frac{1}{18}$$

Random Latin Squares

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Main Results

Problem/Goal

As $n \to \infty$, we want to upper bound the smallest $m \in [n]$ for which L contains no Latin subsquares of order greater than m.

Theorem 1

Let L be a random Latin square. For sufficiently large n, the probability that L contains an order-m Latin subsquare for $m \ge C\sqrt{n \log n}$ tends to 0.

Latin Rectangles

A $k \times n$ Latin rectangle is a $k \times n$ array with n symbols such that each row contains each symbol exactly once and each column contains each symbol at most once.

Consider a partial Latin rectangle to be a $k \times n$ array where each cell is empty or contains one of *n* symbols satisfying the Latin property. We call a partial Latin rectangle C-row compact if in every row, the number of entries with a symbol is at most C.

Theorem 2

For any $\epsilon > 0$, there exists an $\alpha > 0$ such that the following holds for all sufficiently large n: Let P be a αn -row compact partial $k \times n$ Latin rectangle with ℓ nonempty entries, where $k \leq \alpha n$. Then given a random $k \times n$ Latin rectangle L,

If a random $k \times n$ Latin rectangle has some property with high enough probability, then so will the first krows of a random order-n Latin square. Hence, we proved Theorem 2 and used it to prove Theorem 1.

 $\left(\frac{1-\epsilon}{n}\right)^{\ell} \leq \mathbb{P}(P \subset L) \leq \left(\frac{1+\epsilon}{n}\right)^{\ell}.$

Graph Theory Equivalence

Order-n Latin squares correspond to edge-colorings of the complete bipartite graph $K_{n,n}$ with *n* colors, where each color class is a matching.



order-3 Latin square

One tool we used in proving Theorem 2 is called **switch**ings. We consider the set of graphs, A, that contain an edge *e* colored with a specific color and another set of graphs, B, that have mostly the same pattern except that they do not have edge e. We estimate the sizes of A and B by counting the number of switches between the sets. A **switch** is possible if we can switch an edge with another edge of the same color.



The figures above are an example of a 4-cycle switch for the case of symbol 2 appearing in the first column and second row. We use "4-cycle switches" to find a lower bound, where a "cycle" alternates between an edge that exists in the structure and an "edge" that is not present in the structure (denoted by a dashed line). An almost identical argument works using "6-cycle switches" to find an upper bound.

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edge-coloring of $K_{3,3}$

Switchings



switching out

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