## Random Latin Squares

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## Background

## Latin Squares and Subsquares

An order- $n$ Latin square is an $n \times n$ array of $n$ symbols, such that each row and column contains each symbol exactly once.
An order-m subsquare of an order- $n$ Latin square is the $m \times m$ array induced by some set of (not necessarily adjacent) $m$ rows and $m$ columns.
Let $\mathbf{L}$ be an order- $n$ Latin square chosen uniformly at random.

| 1 | 2 | 3 | 4 |
| :--- | :--- | :--- | :--- | :--- |
| 2 | 3 | 4 | 1 |
| 3 | 4 | 1 | 2 |
| 4 | 1 | 2 | 3 |

order-4 Latin square with order-2 Latin subsquare

$$
\begin{array}{|l|}
\hline \text { Conjecture (McKay and Wanless, 1999) } \\
\text { - As } n \rightarrow \infty \text {, the expected number of order-3 Latin subsquares of } \\
\mathbf{L} \text { tends to } \frac{1}{18} \text {. } \\
\text { - As } n \rightarrow \infty \text {, the probability } \mathbf{L} \text { contains a Latin subsquare of order } \\
\text { greater than } 3 \text { tends to zero. } \\
\hline \text { Probabilistic Heuristic } \\
\hline \text { For tuples }(i, j, k) \text { and }\left(i^{\prime}, j^{\prime}, k^{\prime}\right), \\
\mathbb{P}\left(\mathbf{L}_{\mathbf{i j}}=k \text { and } \mathbf{L}_{\mathbf{i}^{\prime} j^{\prime}}=k^{\prime}\right) \approx \mathbb{P}\left(\mathbf{L}_{\mathbf{i j}}=k\right) \mathbb{P}\left(\mathbf{L}_{\mathbf{i}^{\prime} j^{\prime}}=k^{\prime}\right)=\frac{1}{n^{2}} \\
\text { Essentially, for large enough } n \text { and a small enough set } \\
\text { of entries, we can approximate these events as inde- } \\
\text { pendent. }
\end{array}
$$

Example: There are $\binom{n}{3}^{3}$ ways to choose the rows, columns, and symbols for an order- 3 subsquare and $3!\times 2$ ways for it to be Latin. Hence, the expected number of order-3 Latin subsquares in $L$ tends to

$$
\lim _{n \rightarrow \infty} \frac{\binom{n}{3}^{3} \cdot 3!\cdot 2}{n^{9}}=\frac{1}{18}
$$

## Main Results

## Problem/Goal

As $n \rightarrow \infty$, we want to upper bound the smallest $m \in[n]$ for which $\mathbf{L}$ contains no Latin subsquares of order greater than $m$.

## Theorem 1

Let $L$ be a random Latin square. For sufficiently large $n$, the probability that $\mathbf{L}$ contains an order- $m$ Latin subsquare for $m \geq C \sqrt{n \log n}$ tends to 0 .

## Latin Rectangles

A $k \times n$ Latin rectangle is a $k \times n$ array with $n$ symbols such that each row contains each symbol exactly once and each column contains each symbol at most once.

Consider a partial Latin rectangle to be a $k \times n$ array where each cell is empty or contains one of $n$ symbols satisfying the Latin property. We call a partial Latin rectangle $C$-row compact if in every row, the number of entries with a symbol is at most $C$.

## Theorem 2

For any $\epsilon>0$, there exists an $\alpha>0$ such that the following holds for all sufficiently large $n$ : Let $P$ be a $\alpha n$-row compact partial $k \times n$ Latin rectangle with $\ell$ nonempty entries, where $k \leq \alpha n$. Then given a random $k \times n$ Latin rectangle $L$,

$$
\left(\frac{1-\epsilon}{n}\right)^{\ell} \leq \mathbb{P}(P \subset L) \leq\left(\frac{1+\epsilon}{n}\right)^{\ell}
$$

If a random $k \times n$ Latin rectangle has some property with high enough probability, then so will the first $k$ rows of a random order- $n$ Latin square. Hence, we proved Theorem 2 and used it to prove Theorem 1.

## Graph Theory Equivalence

Order- $n$ Latin squares correspond to edge-colorings of the complete bipartite graph $K_{n, n}$ with $n$ colors, where each color class is a matching.

| $G$ | $R$ |
| :--- | :--- |
| $R$ | $B$ |
| $B$ | $G$ |

order-3 Latin square

## Switchings

One tool we used in proving Theorem 2 is called switchings. We consider the set of graphs, A, that contain an edge $e$ colored with a specific color and another set of graphs, B, that have mostly the same pattern except that they do not have edge $e$. We estimate the sizes of $A$ and $B$ by counting the number of switches between the sets. A switch is possible if we can switch an edge with another edge of the same color.

switching in

switching out

The figures above are an example of a 4-cycle switch for the case of symbol 2 appearing in the first column and second row. We use "4-cycle switches" to find a lower bound, where a "cycle" alternates between an edge that exists in the structure and an "edge" that is not present in the structure (denoted by a dashed line). An almost identical argument works using "6-cycle switches" to find an upper bound.

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