Brill-Noether Theory of Fish Graphs

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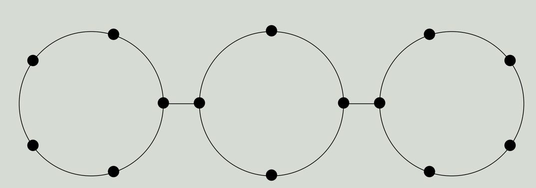
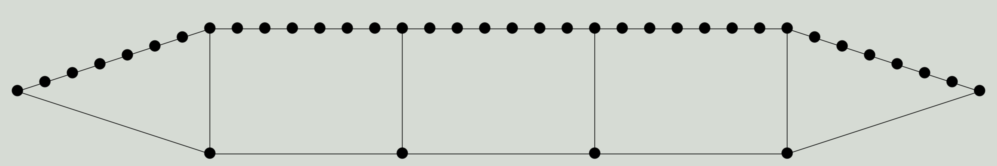
General Question:

When do finite graphs behave like general algebraic curves?

The only nontrivial known family of arbitrary genus where this analogy holds is chains of loops [Pflueger (2017)].



We consider the family of graphs $F_{n,g}$, which are bridgeless, unlike chains of loops.



$F_{7,5}$

Background: How to do algebraic geometry on graphs

Definition: A **divisor** on a graph G is an assignment of an integer to each vertex, representing numbers of chips.

Definition: The **degree** of a divisor is the sum of its values at all vertices of G.

Definition: **Firing a vertex** gives one chip to each of its neighbors. This preserves degree.

Definition: Two divisors are **linearly equivalent** if they can be made equal by firing vertices.

Definition: A divisor is **effective** if it has a nonnegative value at every vertex.

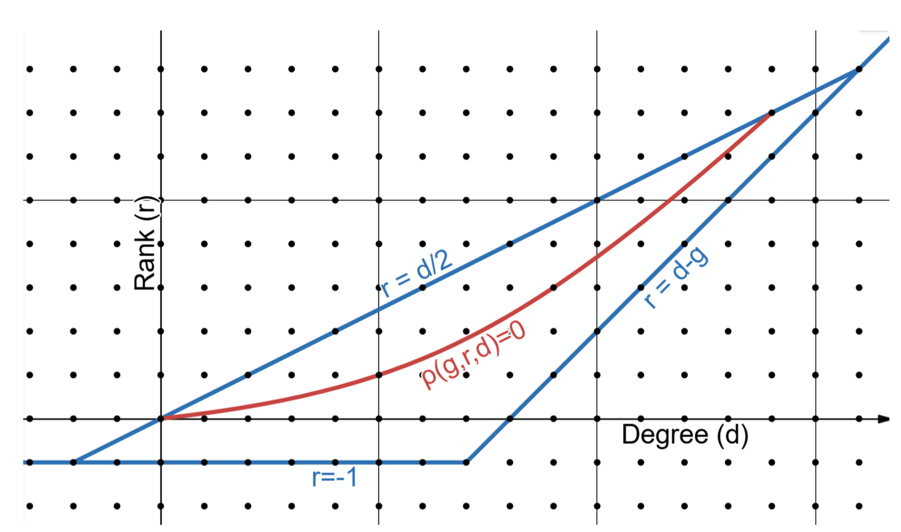
Definition: The **rank** $r(D)$ of a divisor D is -1 if the divisor isn't equivalent to an effective divisor. Otherwise, it's the largest amount of chips that can be removed arbitrarily while still guaranteeing that the divisor is equivalent to an effective divisor.

Definition: The **genus** g of a graph is the number of distinct cycles.

Theorem: Riemann-Roch for Graphs [Baker and Norine (2007)]:

$$r(D) - r(K_G - D) = \deg(D) + 1 - g.$$

In particular, if we graph the rank and degree of divisors on a graph G of genus g, then they must lie inside the blue triangle.



Definition: The **Brill-Noether number** of a divisor D of rank r and degree d is given by

$$\rho(g, r, d) = g - (r + 1)(g - d + r).$$

From classical algebraic geometry there exists a divisor of rank r and degree d if and only if $\rho(g, r, d) \geq 0$, i.e. (r, d) is on the bottom right of the red curve.

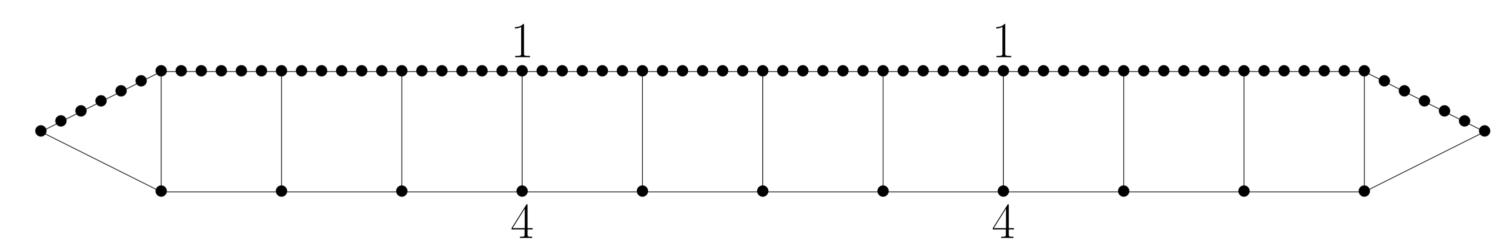
Specific Question:

For a given n and g, is it true that there exists a divisor D of rank r and degree d on $F_{n,g}$ if and only if $\rho(g, r, d) \geq 0$?

Theorem: If $\rho(g, r, d) \geq 0$, then there exists a divisor D on $F_{n,g}$ with $r(D) = r$ and $\deg D = d$.

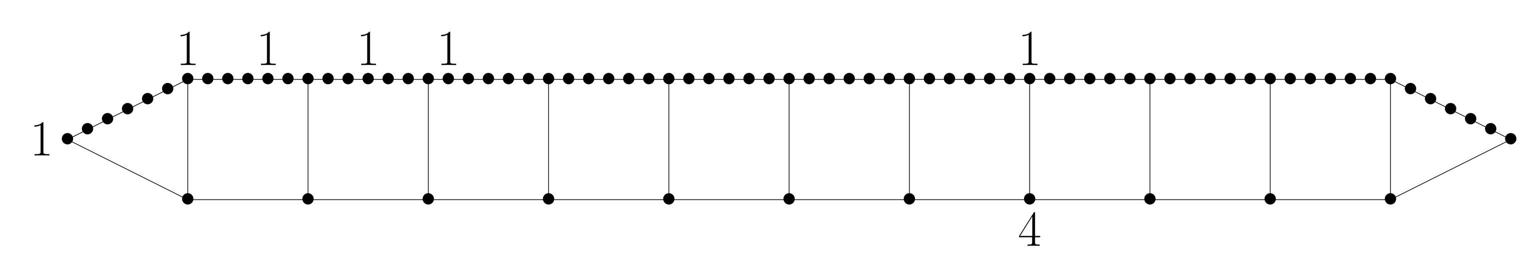
Remark: It suffices to consider the $\rho(g, r, d) = 0$ case and only removing chips from the bottom vertices of the graph.

Example: Genus 12, rank 2, and degree 10.



Lemma: The rank is equal to the number of pillars, i.e. the number of vertical edges with chips on them when written in the original form.

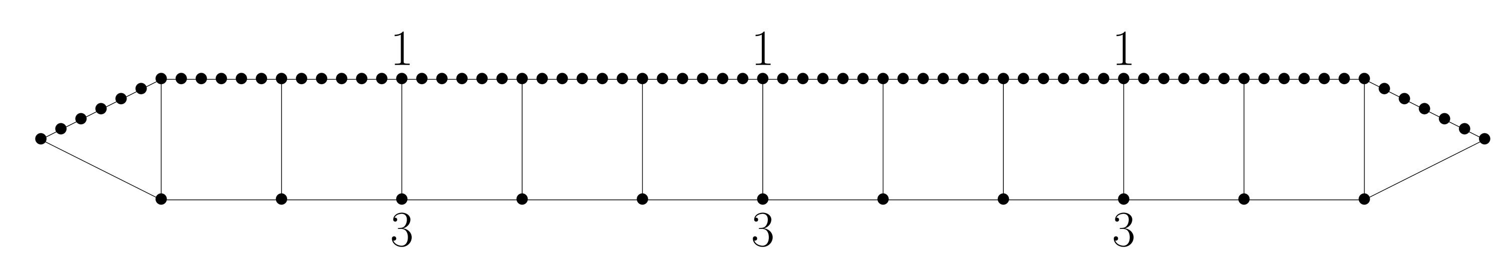
For example, we can perform firing moves on the left pillar to cancel a debt on the leftmost vertex as follows.



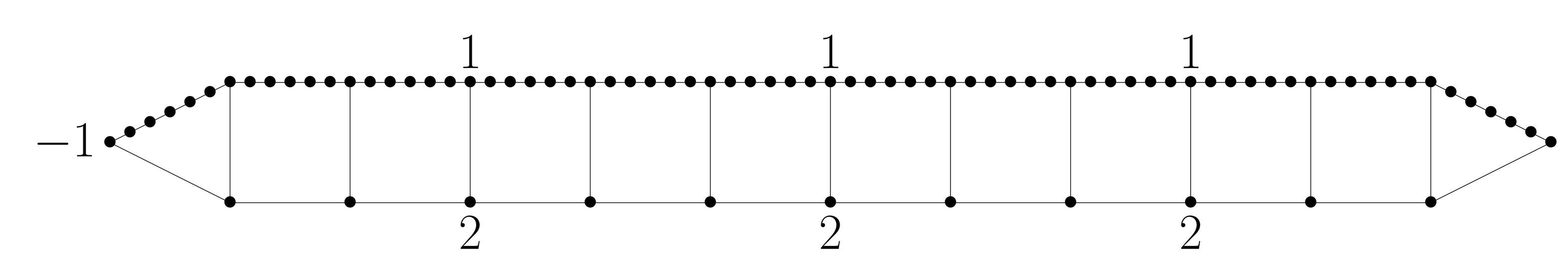
The right pillar can cancel another debt on the right side, or help on the left.

Remark: $\rho(g, r, d) = 0 \iff d = r(\frac{g}{r+1} + 1)$, so this divisor satisfies $\rho = 0$.

Example: Genus 12, rank 3, and degree 12.



To upper bound the rank, we can remove four chips as follows:



The resulting divisor is not equivalent to an effective divisor. Hence the rank is at most three.

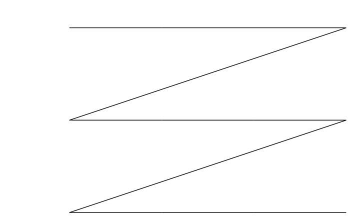
Looking ahead: Enumerating Special Divisors

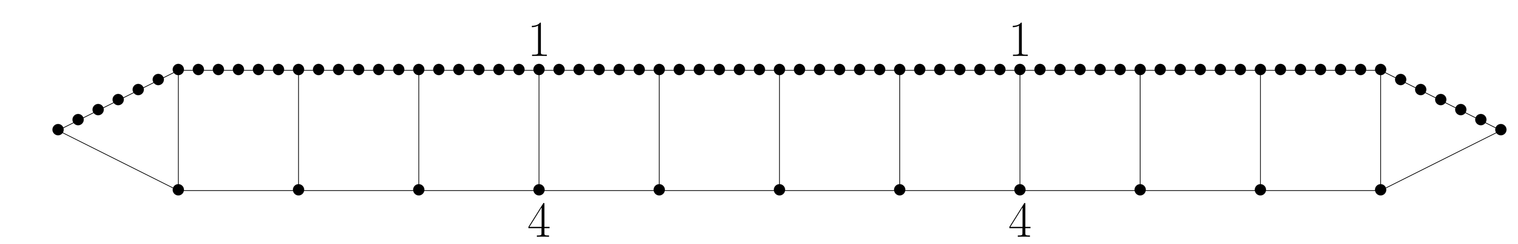
From classical algebraic geometry and the chain of loops case, we expect the divisors of degree d and rank r satisfying $\rho(g, r, d) = 0$ to be in bijection with standard Young tableaux of shape $(r + 1) \times (g - d + r)$, up to linear equivalence.

Definition: A **standard Young tableau** of shape $a \times b$ is a grid with a rows and b columns filled with the numbers 1 through ab, so that the numbers increase down and to the right.

Given a standard Young tableau, we can trace out the path given by following the numbers in order. At each step of the path, we place chips on the next cycle according to displacement vector.

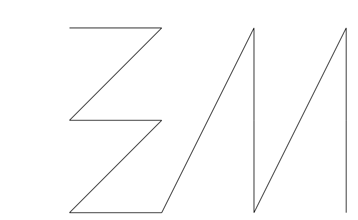
1	2	3	4
5	6	7	8
9	10	11	12

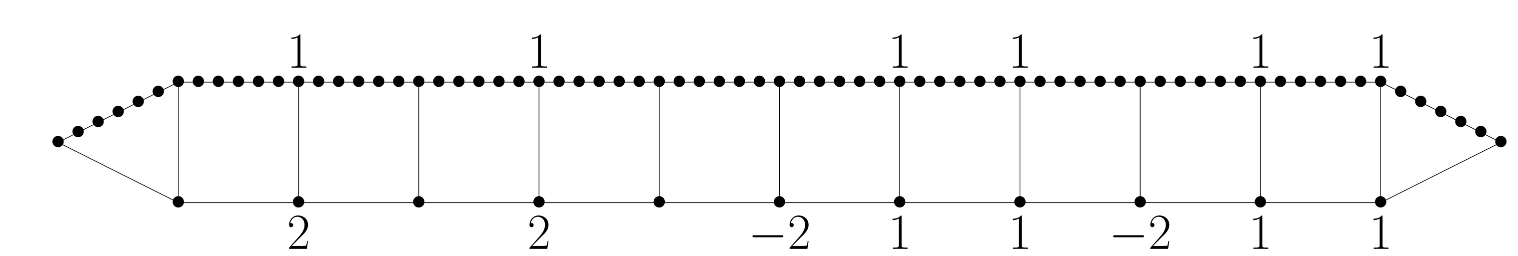




This construction gives lots of new divisors with $\rho = 0$!

1	2	7	10
3	4	8	11
5	6	9	12





The above divisor happens to be built out of smaller rank two divisors, so it has rank two itself. This corresponds to the Young tableau having two plus one rows.

Conjectures and Questions

1. A standard Young tableau of shape $a \times b$ gives a divisor with rank $a - 1$ and $\rho = 0$.
2. For $n >> g$, the map from standard Young tableaux to divisors of rank $a - 1$ and $\rho = 0$ is a bijection.
3. Can we extend this construction to divisors with $\rho \neq 0$ and use that to show that there are no divisors with $\rho < 0$?

References

Baker, M. and Norine, S. (2007). Riemann–Roch and Abel–Jacobi theory on a finite graph. *Advances in Mathematics*, 215(2):766–788.

Pflueger, N. (2017). Special divisors on marked chains of cycles. *Journal of Combinatorial Theory, Series A*, 150:182–207.