Prym varieties of folded k-gonal chains of loops

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Introduction: Chip-Firing

Let Γ be a metric graph. A *divisor* D on Γ is an element of the free abelian group on the points in Γ. This can be thought of as putting an integer number of chips on a finite number of points in Γ.

A *chip-firing move* on a divisor D is a movement of the chips along the graph such that net momentum across each cycle in Γ is 0.

Chip-firing defines an equivalence relation: 2 divisors D and D' are equivalent if there exists a series of chip-firing moves that takes D to D'.

The *degree* of a divisor is the total number of chips of the divisor. An *effective* divisor is a divisor where every point has a nonnegative number of chips.

The *rank* of a divisor $r(D)$ is the largest nonnegative integer r such that $D-E$ is equivalent to an effective divisor for all effective divisors E of degree r. If no such r exists, then the divisor has rank -1.

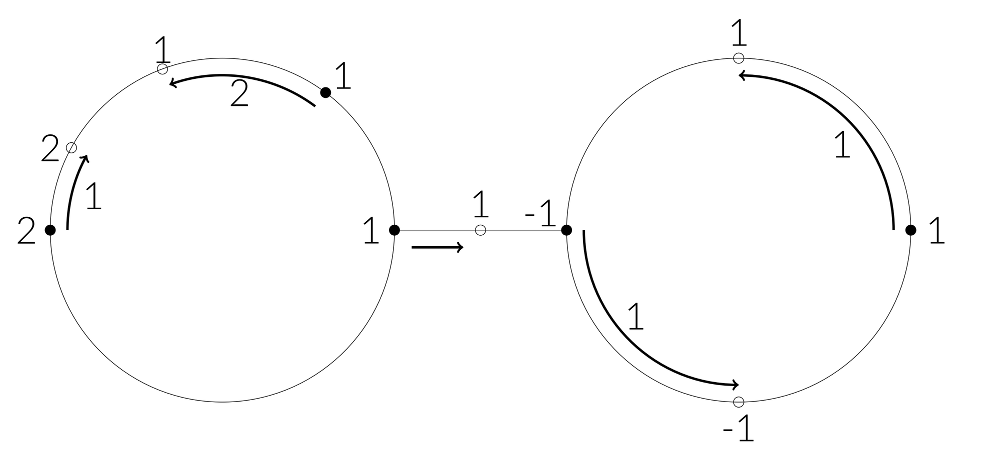

Figure 1: Examples of chip-firing moves using the momentum definition: a solid dot is the starting position, and an open one is the ending position. Note that the bridge is not a part of any cycle, so chips may be moved freely on that edge.

Prym Divisors

We wish to look at a certain type of graph: the *chain of g loops*, consisting of g cycles connected together by bridges. The *gonality* of a chain of loops is some fixed integer k such that on each interior loop, the top arc has length $k-1$ and the bottom arc has length 1. For an individual loop, k is referred to as its *torsion*.

For a metric graph Γ, the *canonical divisor* K_Γ is defined by placing $\mathrm{val}(v)-2$ chips at each point for all points v in Γ. In the case of the chain of loops, the canonical divisor has a chip at each endpoint of each bridge.

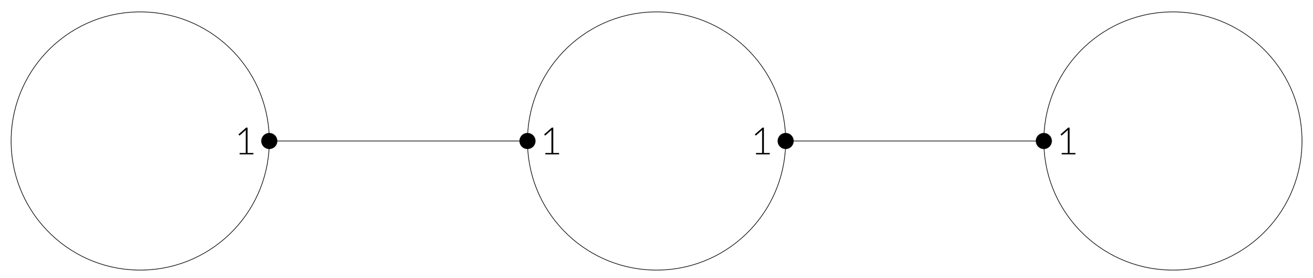

Figure 2: The canonical divisor on a chain of 3 loops.

A *folded chain of loops* (see Figure 4) is a specific double cover $\pi : \tilde{\Gamma} \to \Gamma$ such that:

- Γ is a chain of g loops $\gamma_1, \ldots, \gamma_g$.
- The preimage of each loop $\gamma_1, \ldots, \gamma_{g-1}$ consists of two disjoint loops.
- The preimage of γ_g consists of a single loop.

For the folded k-gonal chain of loops, every loop has torsion k with the exception of the preimage of the g^{th} loop, which has torsion 2.

A *Prym divisor* is a divisor that maps down to K_Γ. We study the *Prym-Brill-Noether locus* of a folded k-gonal chain of loops, which is a subset of Prym divisors of at least a certain rank, denoted by $V_r(\pi)$. More specifically, for a fixed rank r, this is the set of Prym divisors $\tilde{D}$ on $\tilde{\Gamma}$ such that

- $r(\tilde{D}) \geq r$,
- $r(\tilde{D}) \equiv r \mod 2$,
- $\pi_*(\tilde{D}) = K_\Gamma$,

where r is the rank function and π_* is the induced map on divisors. It is worth noting that since the degree of K_Γ is $2g-2$, we have that the degree of Prym divisors is always $2g-2$. It is known that $V_0(\pi)$ and $V_{-1}(\pi)$ are disjoint and each isomorphic to $(\mathbb{R}/\mathbb{Z})^{g-1}$, and $V_{r+2}(\pi)$ is a strict closed subset of $V_r(\pi)$ when $V_r(\pi)$ is nonempty.

Prym Tableaux

A *Prym tableau* t is an $(r+1)\times(r+1)$ Young tableau on $2g-1$ symbols that satisfies the following conditions:

- *The Young condition*: Every row and column must be strictly increasing.
- *The displacement condition*: If a symbol m repeats in the tableau, the taxicab distance between any pair of repeats must be a multiple of the torsion of the m^{th} loop.
- *The Prym condition*: If $m = t(a,b) = 2g - t(c,d)$, then $a-b \equiv c-d \mod k$ where k is the torsion of the m^{th} loop.

For example, let us consider the following Prym tableau that corresponds to a subset of divisors on a 4-gonal folded chain of 7 loops, where rank is at least 3. Note that all symbols can appear a multiple of 4 away in the taxicab distance, with the exception being 7: the 7^{th} loop has torsion 2, so the distance between repeats must be multiples of 2.

5		
4	6	
1	2	5

7	9	10	13
5	7	8	12
4	6	7	9
1	2	5	7

Figure 3: An example of a valid Prym tableau for $g=7$, $k=4$, $r=3$. Note the lower triangular tableau on the left completely determines the full square one on the right (see Theorem 1).

Tableaux to Divisors

A valid Prym tableau encodes Prym divisors of rank at least r on a folded k-gonal chain of g loops. To go from tableau to divisor, we do the following:

1. On the first loop, we put -1 chips distance 1 clockwise (CW) away from the right bridge.
2. For symbol $m < g$ in cell (x,y), we put a chip distance $x-y$ counterclockwise away from the left bridge of the m^{th} loop. (For $m=1$, the left bridge is the vertex described in the first step.)
3. For symbol $m > g$ in cell (x,y), we put a chip distance $x-y$ CW away from the right bridge of the m^{th} loop.
4. For r odd, place the chip on the bottom of the g^{th} loop; otherwise place the chip on the top of the loop.
5. If only one of m or $2g-m$ appears in the tableau, it fixes the position of $2g-m$ or m respectively. The unspecified chip is placed so that when mapped down, the divisor is equivalent to K_Γ.
6. If symbols m and $2g-m$ both do not appear in the tableau, then the chip on the m^{th} loop can be placed freely, and the chip on the $2g-m^{\text{th}}$ loop is placed in the corresponding location to map down to K_Γ.

This tableau corresponds to the Prym divisor in Figure 4. We note that since 3 and 11 do not appear in the tableau, the corresponding loops are free, and the chip could be placed anywhere along the loop. We observe that the red arrows describe chip-firing moves that make the resulting divisor equivalent to K_Γ.

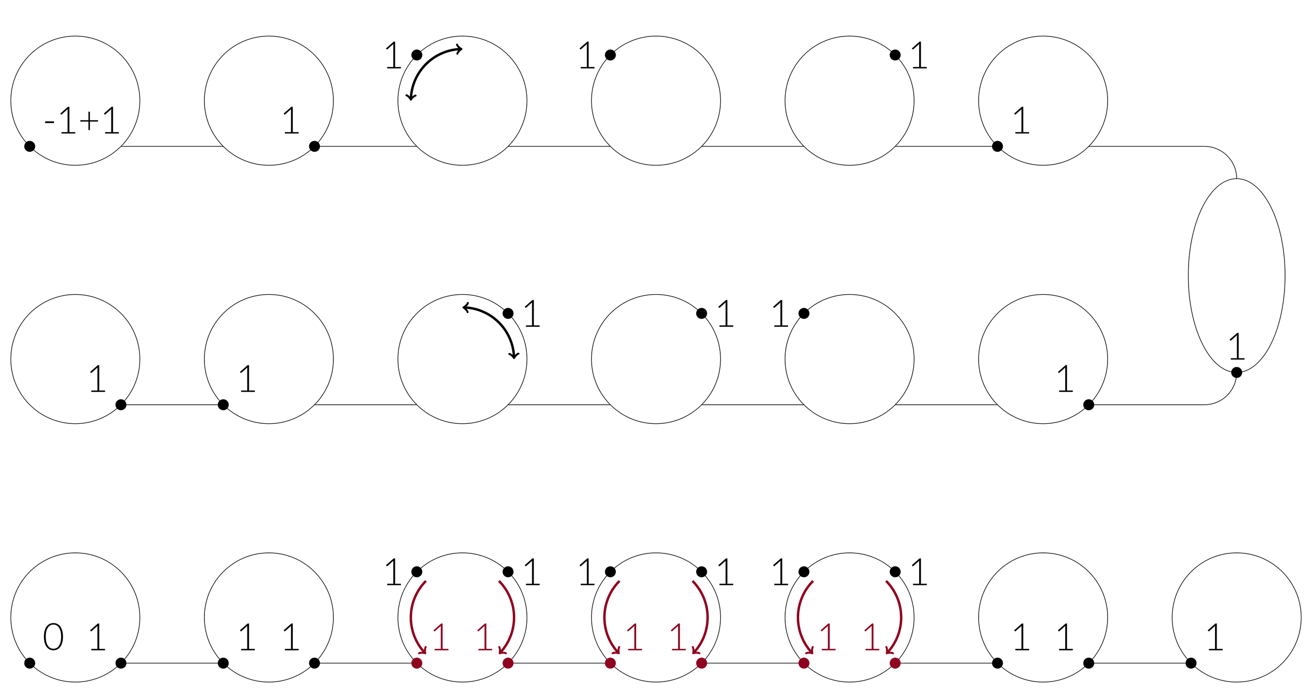

Figure 4: A Prym divisor on the folded 4-gonal chain of 7 loops, and the map down into K_Γ.

Dimension, Codimension, and the Reflection Theorem

Given a Prym tableau t, we define the *codimension* of t to be the number of pairs $(m, 2g-m)$ where $m<g$ and at least one of m or $2g-m$ appears in tableau.

Rather than considering the entire tableau, we prove the following theorem.

Theorem 1: Reflection Theorem

Given an $(r+1)\times(r+1)$ Prym tableau t, there is a lower triangular tableau s of size r that produces a set of divisors such that the divisors described by t is a subset of those described by s.

If the codimension of a Prym tableau is n, it corresponds to a $g-1-n$ dimensional subspace of one of the disjoint copies of $(\mathbb{R}/\mathbb{Z})^{g-1}$. We say a Prym tableau is *maximizing* if it has minimal codimension. Thus, by calculating the codimension of maximizing tableau, we can calculate the dimension of the Prym-Brill-Noether locus.

Topology of Prym-Brill-Noether Loci

Using the reflection theorem, we bound the codimension of maximizing Prym tableau, completing the classification of the dimension of $V_r(\pi)$ that was started by Len and Ulirsch [LU19]:

Theorem 2: Dimension Theorem

The dimension of the Prym-Brill-Noether locus is:

$$\begin{cases} g-1-\binom{r+1}{2} & k > 2r-2 \\ g-1-\left(\frac{rk}{2}-\frac{k^2}{8}+\frac{k}{4}\right) & k \leq 2r-2 \text{ even} \\ g-1-\left(\frac{rk}{2}+\frac{r}{2}-\frac{k^2-1}{8}\right) & k \leq 2r-2 \text{ odd} \end{cases}.$$

Additionally, we show the following:

Theorem 3: Path Connected and Pure Dimensional

$V_r(\pi)$ is pure dimensional. When $\dim(V_r(\pi)) > 0$, then $V_r(\pi)$ is path connected.

For $k > 2r-2$, we can use hook-length to enumerate the points of $V_r(\pi)$ when the dimension is 0. For k even, repeats in the tableau make it harder to count, and thus we cannot use the hook-length formula. Instead, we form a bijection between tableaux and lattice paths, which gives us a formula for enumeration, using [1, Theorem 10.18.6].

Finally, we look at $V_r(\pi)$ when it is 1-dimensional:

Theorem 4: Homology when $\dim(V_r(\pi)) = 1$

For $k > 2r-2$, when $\dim(V_r(\pi)) = 1$, the rank of the first homology of $V_r(\pi)$ is

$$\frac{rf^\lambda(n+1)}{2}+1,$$

where $n = \binom{r+1}{2}$. This is a special case of [2, Theorem 2.9].
For $k=2$, the genus of the resulting graph is

$$r+1.$$

For $k=4$, the genus is

$$2^{r-1}(3r-2)+1.$$

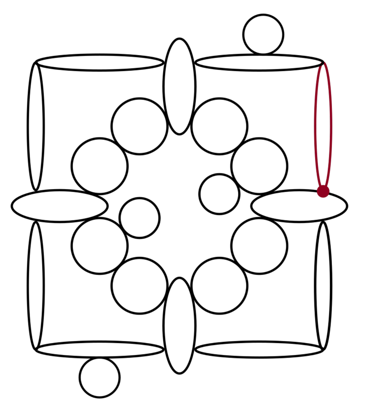

Figure 5: The 1-dimensional $V_3(\pi)$ for $g=7$, $k=4$. The highlighted cycle corresponds to the tableau in Figure 3, and the highlighted vertex corresponds to the Prym divisor in Figure 4.

Open Problems

- When k is odd, the number of points of a 0-dimensional locus is still unknown.
- For dimension greater than 1, topological aspects such as the Euler characteristic and rank of homology groups are still unknown.
- A generalization of the tableau method to other covers of higher degree is unknown.

Acknowledgement

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