Prym varieties of folded k-gonal chains of loops

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Divisors on chains of loops

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- A *chain of loops* is a metric graph Γ consisting of cycles connected together by bridges. The number of cycles (or equivalently, the genus) is denoted by g.

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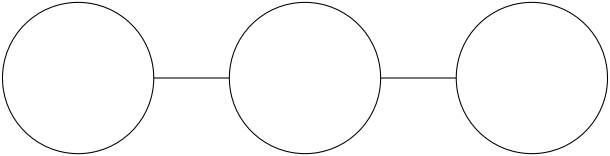

Figure: A chain of 3 loops

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- Without loss of generality, let the bottom arc of each loop have length 1.

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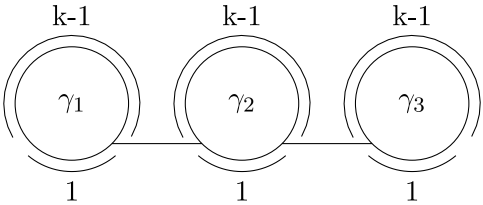

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- The *degree* d of a divisor D is the sum of the chips in D.

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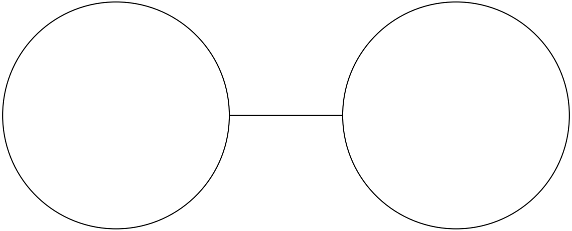

Figure: A divisor of degree 4 on the chain of 2 loops

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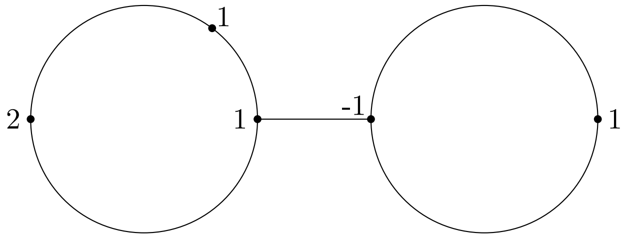

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- Chip-firing defines an equivalence relation: two divisors D and D' are equivalent just if there exists a series of chip-firing moves that takes D to D'.

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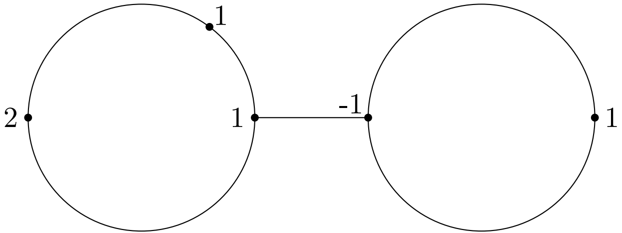

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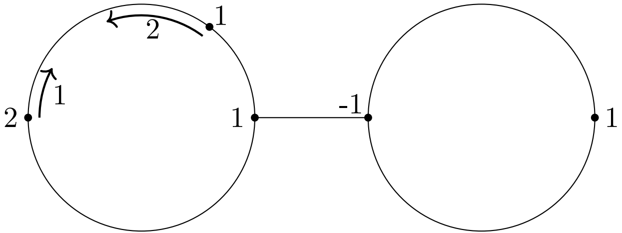

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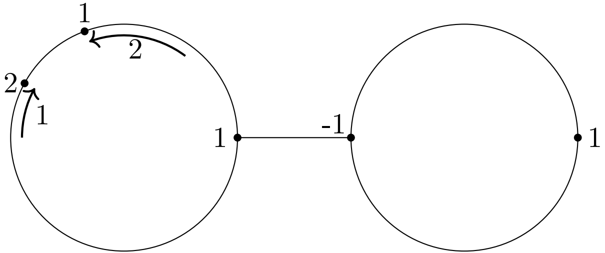

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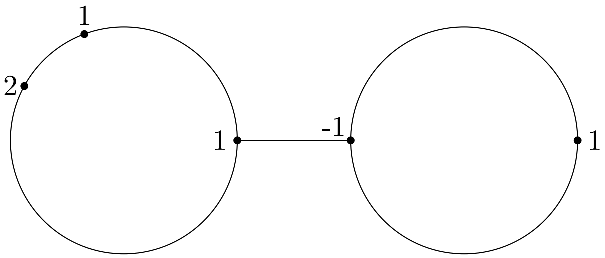

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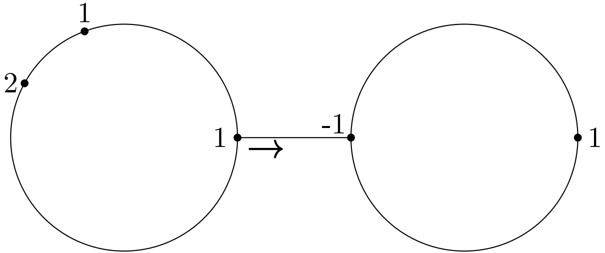

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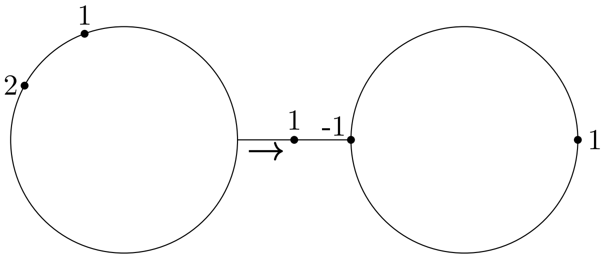

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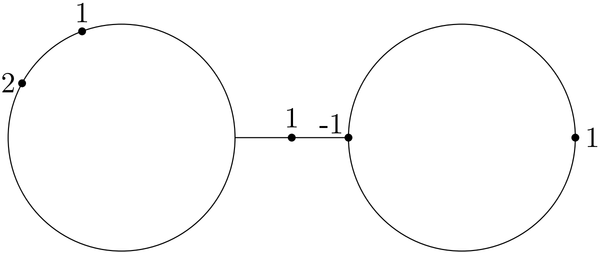

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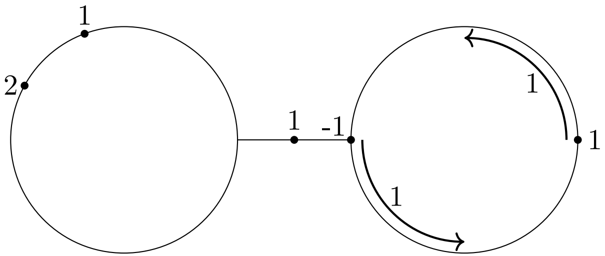

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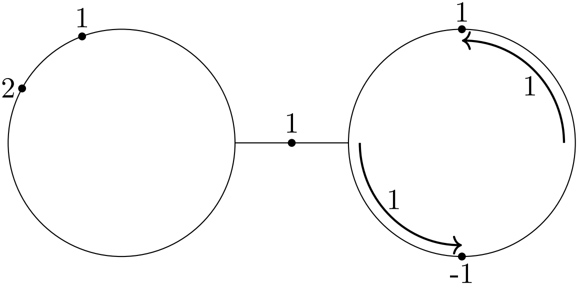

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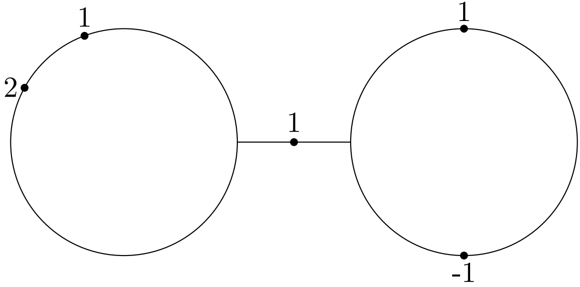

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- Brill-Noether theory classifies the divisors on a metric graph of degree d and rank at least r.

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- Certain results proved here in the tropical case have implications in the algebraic case.

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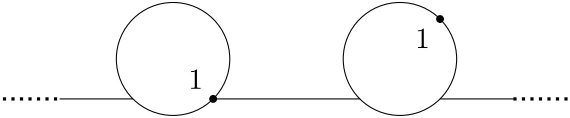

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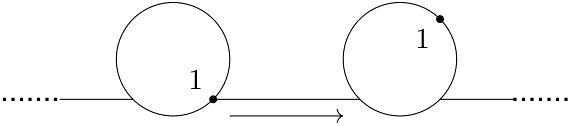

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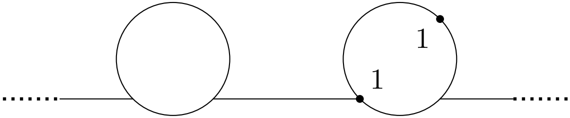

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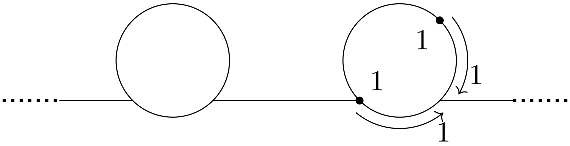

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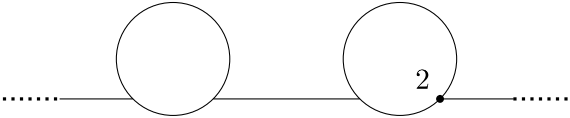

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- We are interested in a specific double cover of the chain of loops called the *folded chain of loops*.

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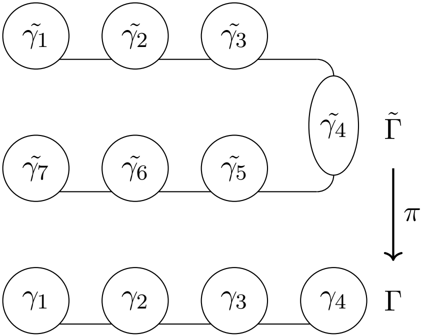

Figure: The folded k-gonal chain of 4 loops

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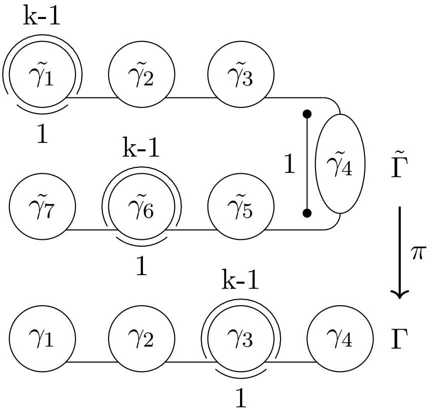

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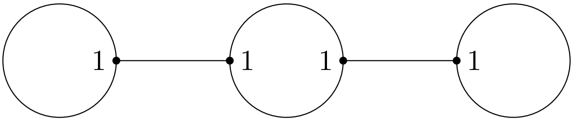

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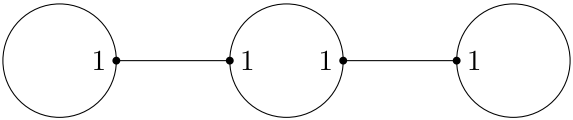

- The *Prym variety*—the space of all classes of Prym divisors—has the structure of two disjoint copies of the $(g-1)$-dimensional torus.

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- V^{-1} and V^0 constitute the two disjoint copies of $(g-1)$-dimensional tori, and contain the odd- and even-ranked divisors, respectively.

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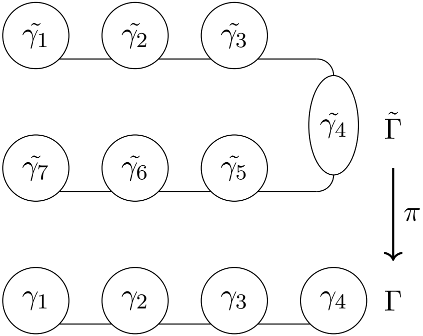

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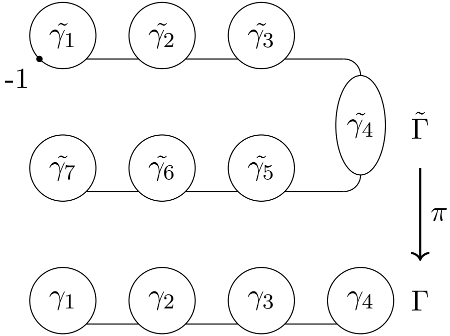

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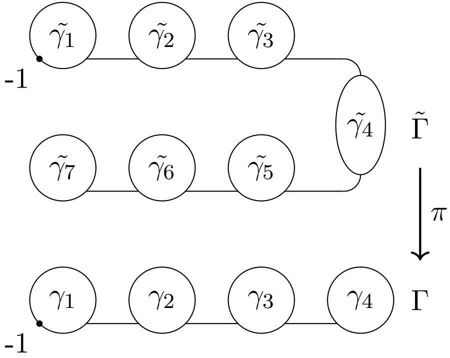

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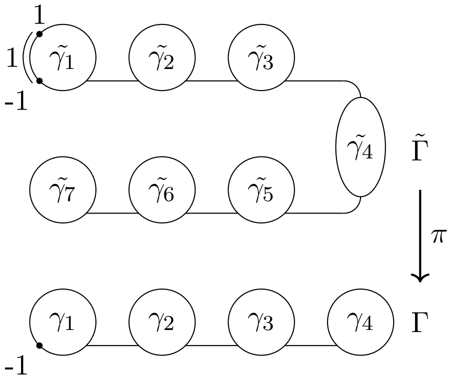

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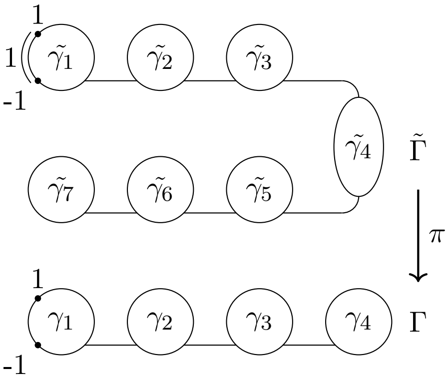

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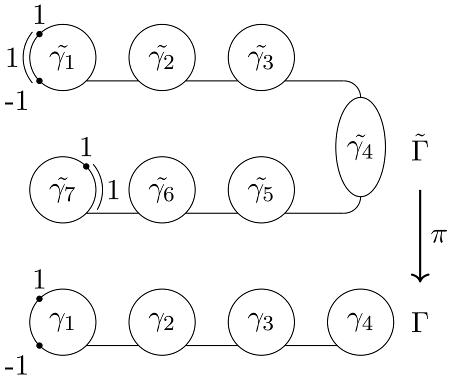

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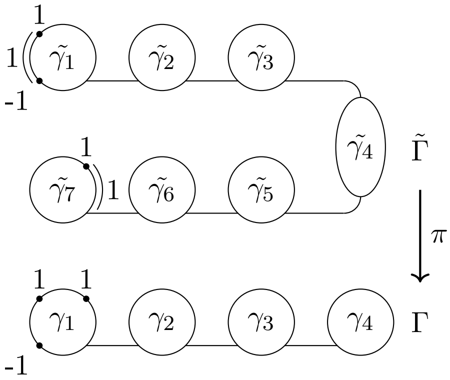

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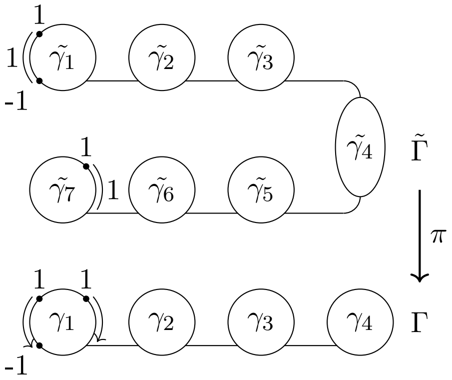

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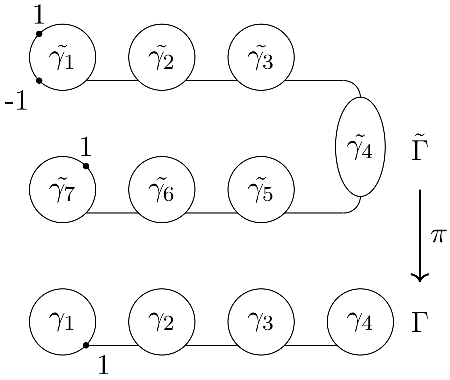

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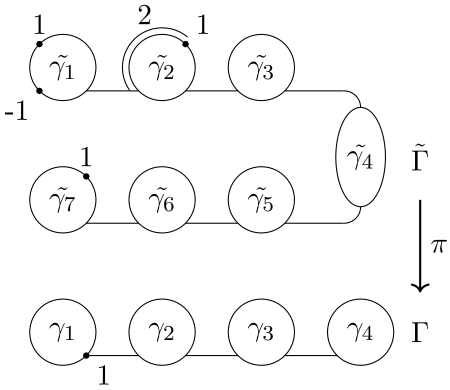

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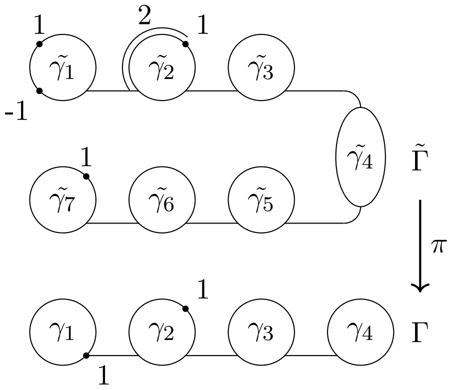

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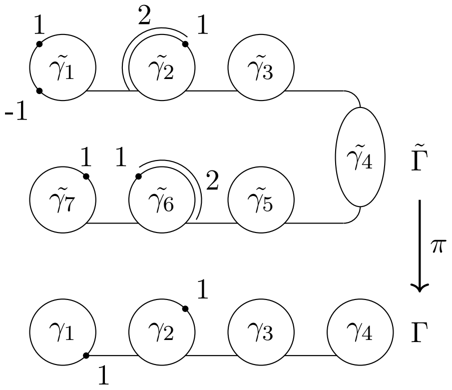

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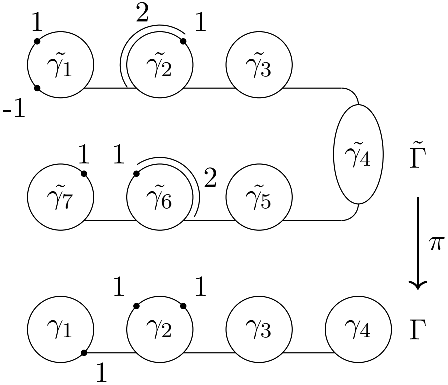

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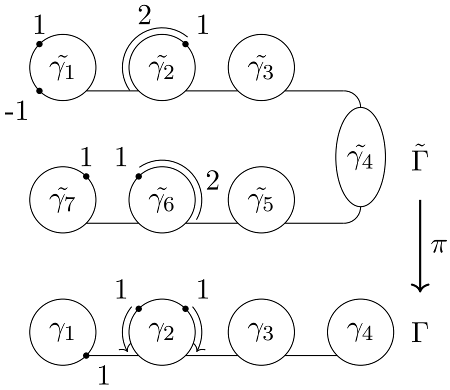

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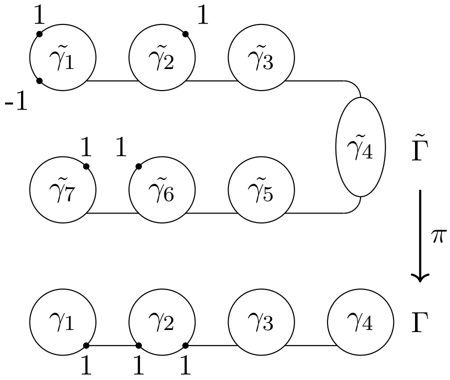

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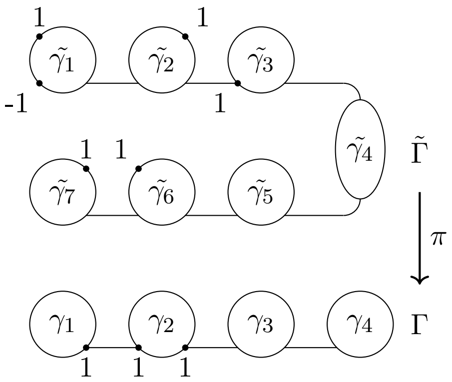

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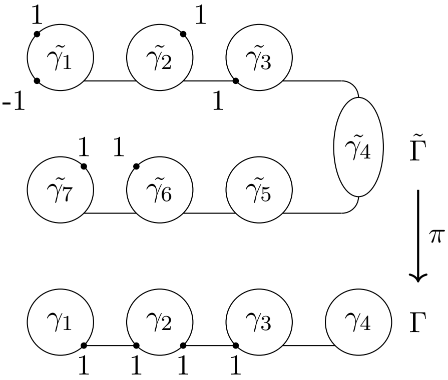

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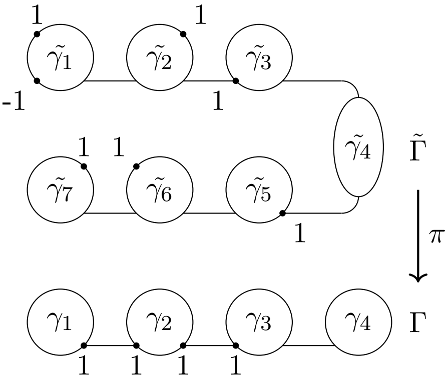

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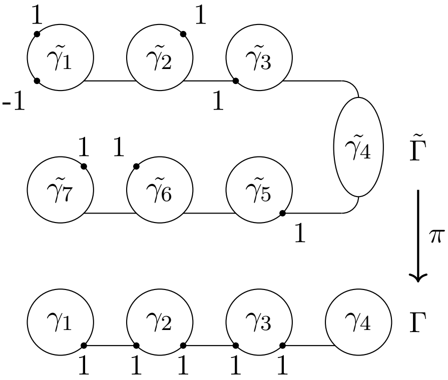

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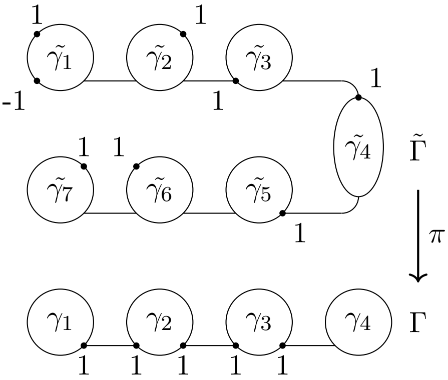

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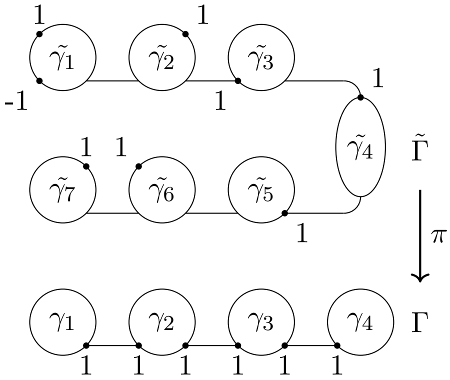

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- A *Prym tableau* for a given rank r is a square grid of length $r+1$ where each cell in the grid is filled out with the first $2g-1$ integers, called *symbols*.
- We can index through the tableau by diagonals modulo k: For a given symbol n with torsion k, the i-th diagonal is all cells (x, y) such that $x - y \equiv i \pmod{k}$.
  - The *standard condition*: every row and column must be strictly increasing.
  - The *displacement condition*: If symbol n repeats in the tableau, then all repeats must be in the same diagonal mod k.
  - The *Prym condition*: If symbols n and $2g-n$ both appear in the tableau, they must be in the same diagonal mod k.

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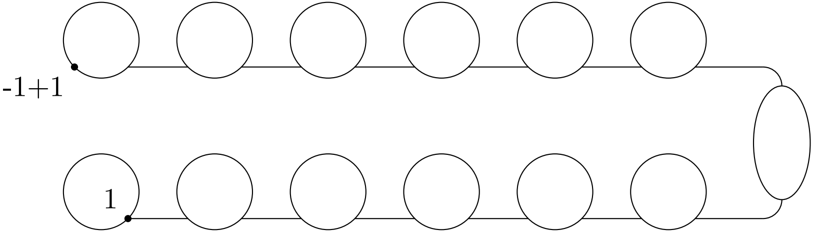
  - For g, if $x - y$ is even, then it goes on the top vertex, otherwise it goes on the bottom vertex

Prym tableaux

7	9	10	13
5	7	8	12
4	6	7	9
1	2	5	7



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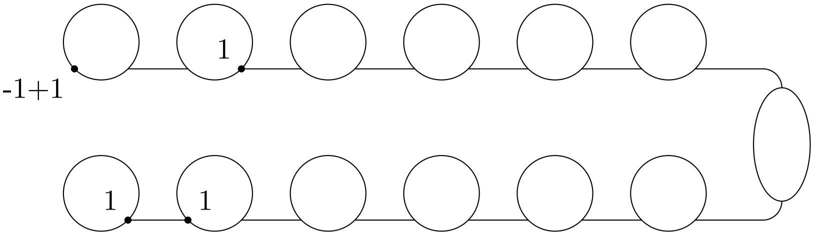

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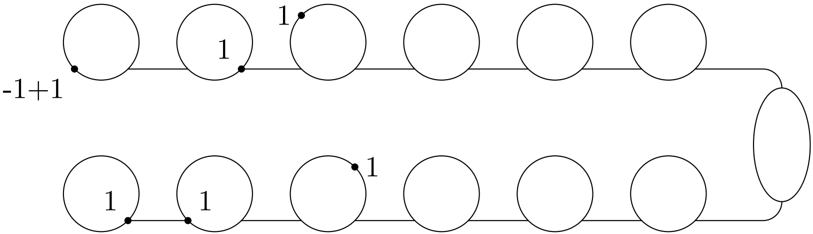

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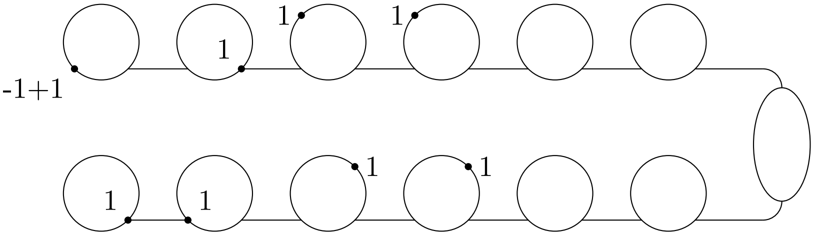

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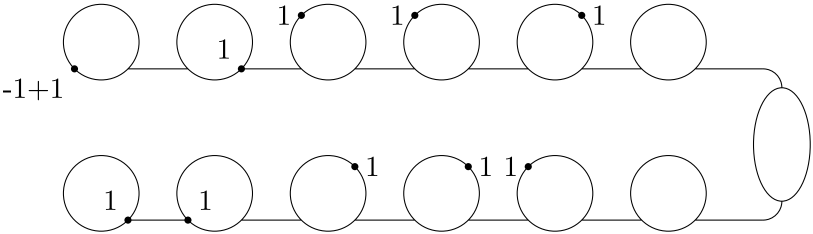

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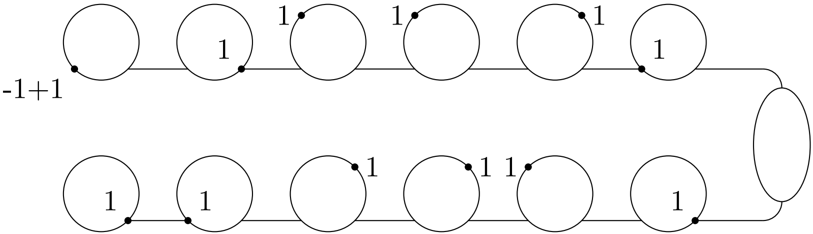

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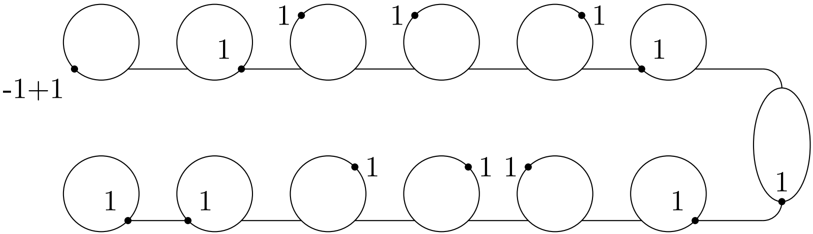

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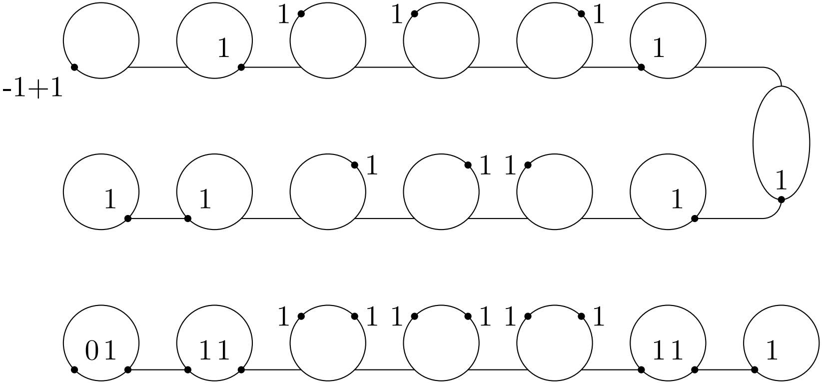

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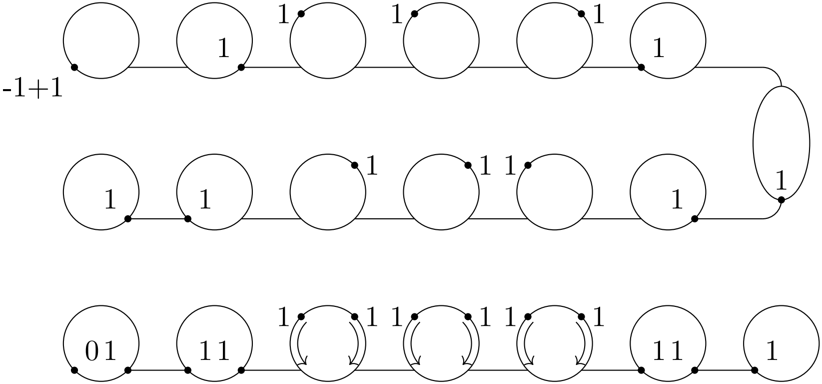

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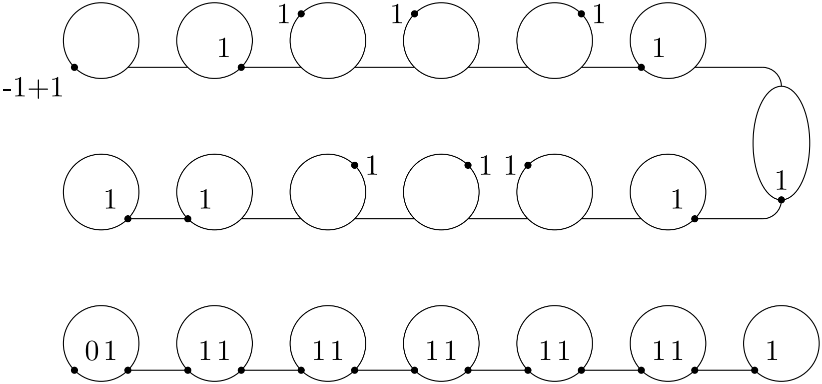

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Theorem

$\dim(V^r) = g - 1 - n$, *where*

$$n = \begin{cases} \binom{r+1}{2} & \text{if } r \le l \\ \binom{l+1}{2} + l(r-l) & \text{if } r > l \end{cases}, \tag{1}$$

and where $l = \lceil \frac{k}{2} \rceil$.

Tropological Results

Theorem

V^r is pure-dimensional.

Theorem

If $\dim(V^r) > 0$, then V^r is path-connected.

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- When k is even, we can create a bijection with a lattice path enumeration problem.
- The cardinality is still unknown for k odd.

1-dimensional loci

- Every 1-dimensional locus is a collection of circles wedged together in some way.

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When $\dim(V^r(\pi)) = 1$, *the rank of the first homology of* $V^r(\pi)$ *is:*

$$\begin{cases} \frac{rf^{\lambda}\left(\binom{r+1}{2}+1\right)}{2} + 1 & k > 2r - 2 \\ r + 1 & k = 2 \\ 2^{r-1}(3r-2) + 1 & k = 4 \end{cases}$$

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- It is unknown for other values of k.

Future work

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- Study tropological properties of V^r for different covering maps (snake of loops, tree of loops, etc.)
- Strengthen the connection to Prym divisors on algebraic varieties.

Thank you!