Prym varieties of folded $k$-gonal chains of loops

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Divisors on chains of loops

- Given a graph $G$, a *metric graph* is the metric space obtained by assigning to each edge of $G$ a closed interval and gluing any two of these intervals together at their endpoints just if their corresponding edges meet at that vertex.
Divisors on chains of loops

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- A chain of loops is a metric graph $\Gamma$ consisting of cycles connected together by bridges. The number of cycles (or equivalently, the genus) is denoted by $g$. 
Divisors on chains of loops

Figure: A chain of 3 loops
Divisors on chains of loops

- The *torsion* of a loop is the ratio of the length of the loop to the length of its bottom arc.
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Divisors on chains of loops

- The *torsion* of a loop is the ratio of the length of the loop to the length of its bottom arc.
- The chain of loops has *gonality* $k$ if all of its loops have torsion $k$.
- Without loss of generality, let the bottom arc of each loop have length 1.
Divisors on chains of loops

Figure: A $k$-gonal chain of 3 loops
Divisors on chains of loops

- A divisor (or chip configuration) $D$ on $\Gamma$ is an element of the free abelian group on the set of points of $\Gamma$. 
Divisors on chains of loops

- A *divisor* (or *chip configuration*) $D$ on $\Gamma$ is an element of the free abelian group on the set of points of $\Gamma$.
- The *degree* $d$ of a divisor $D$ is the sum of the chips in $D$. 
Divisors on chains of loops

Figure: A divisor of degree 4 on the chain of 2 loops
Divisors on chains of loops

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A chip-firing move is a movement of chips in $D$ such that the “net momentum” on each cycle in $\Gamma$ is zero.
Divisors on chains of loops

- A *chip-firing move* is a movement of chips in $D$ such that the “net momentum” on each cycle in $\Gamma$ is zero.
- Chip-firing defines an equivalence relation: two divisors $D$ and $D'$ are equivalent just if there exists a series of chip-firing moves that takes $D$ to $D'$. 
Figure: Examples of valid chip-firing moves given a divisor of degree 4 on a chain of 2 loops.
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Divisors on chains of loops

Every divisor class has a unique representative for which each loop has exactly one chip, except possibly one loop which contains a fixed point \( v \), on which there are an additional \( d - g \) chips.
Divisors on chains of loops

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- The space of equivalence classes of divisors of a fixed degree $d$ has the structure of a $g$-dimensional torus.

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- An effective divisor is one which is nonnegative at every point in $\Gamma$. 

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- An effective divisor is one which is nonnegative at every point in \( \Gamma \).
- The rank of a divisor \( r(D) \) is the largest nonnegative integer \( r \) such that \( D - E \) is equivalent to an effective divisor for all effective divisors \( E \) of degree \( r \). If no such \( r \) exists, then the divisor has rank -1.
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- An effective divisor is one which is nonnegative at every point in $\Gamma$.
- The rank of a divisor $r(D)$ is the largest nonnegative integer $r$ such that $D - E$ is equivalent to an effective divisor for all effective divisors $E$ of degree $r$. If no such $r$ exists, then the divisor has rank -1.
- Brill-Noether theory classifies the divisors on a metric graph of degree $d$ and rank at least $r$. 

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Motivation

- We obtain chains of loops from certain Riemann surfaces via a process known as *tropicalization*.
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- Divisors on tropical varieties (such as metric graphs) are analogous to divisors on algebraic varieties.

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- Divisors on tropical varieties (such as metric graphs) are analogous to divisors on algebraic varieties.
- Divisor classes of rank $r$ on an algebraic curve $C$ are in bijection with maps $C \to \mathbb{P}^r$ up to change of coordinates.
- Certain results proved here in the tropical case have implications in the algebraic case.
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Prym varieties of folded $k$-gonal chains of loops
Rank

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Prym varieties of folded $k$-gonal chains of loops
Rank

\[ \cdots \quad 1 \quad 1 \cdots \]
Prym varieties of folded $k$-gonal chains of loops
Rank

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Prym varieties of folded $k$-gonal chains of loops
A double cover of metric spaces $\pi: \tilde{\Gamma} \to \Gamma$ is a local isometry such that the preimage of each point in $\Gamma$ contains exactly two points.

Prym divisors on folded chains of loops
Prym divisors on folded chains of loops

- A *double cover* of metric spaces \( \pi : \tilde{\Gamma} \to \Gamma \) is a local isometry such that the preimage of each point in \( \Gamma \) contains exactly two points.

- We are interested in a specific double cover of the chain of loops called the *folded chain of loops*.
Prym divisors on folded chains of loops

Figure: The folded $k$-gonal chain of 4 loops
Prym divisors on folded chains of loops

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The map $\pi$ induces a map $\pi_*$ on divisor classes.
Prym divisors on folded chains of loops

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- A Prym divisor is a divisor (class) on $\tilde{\Gamma}$ that maps down to $K_\Gamma$. 
Prym divisors on folded chains of loops

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- The *canonical divisor* $K_\Gamma$ contains $\text{val}(v) - 2$ chips at each point $v \in \Gamma$. 
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The canonical divisor $K_\Gamma$ contains $\text{val}(v) - 2$ chips at each point $v \in \Gamma$.

The Prym variety—the space of all classes of Prym divisors—has the structure of two disjoint copies of the $(g - 1)$-dimensional torus.
The *Prym–Brill–Noether locus* for some fixed $r$, denoted $V^r$, is the set of Prym divisors that satisfy the following conditions:
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$V^{-1}$ and $V^0$ constitute the two disjoint copies of $(g - 1)$-dimensional tori, and contain the odd- and even-ranked divisors, respectively.
Prym divisors on folded chains of loops

\[ \tilde{\gamma}_7 \quad \tilde{\gamma}_6 \quad \tilde{\gamma}_5 \quad \tilde{\gamma}_4 \]

\[ \gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \gamma_4 \]

\[ \tilde{\Gamma} \quad \Gamma \]

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Prym divisors on folded chains of loops

\[ \tilde{\gamma}_1 \rightarrow \tilde{\gamma}_2 \rightarrow \tilde{\gamma}_3 \rightarrow \tilde{\gamma}_4 \]

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\[ \tilde{\Gamma} \]

\[ \Gamma \]

\[ \pi \]

-1

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Prym divisors on folded chains of loops
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\[ \tilde{\gamma}_1 \to \tilde{\gamma}_2 \to \tilde{\gamma}_3 \to \tilde{\gamma}_4 \]

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\[ \Gamma \leftarrow \tilde{\Gamma} \]

\[ \pi \]

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\[ \gamma_1 \rightarrow \gamma_2 \rightarrow \gamma_3 \rightarrow \gamma_4 \rightarrow \tilde{\gamma}_4 \rightarrow \tilde{\gamma}_5 \rightarrow \tilde{\gamma}_6 \rightarrow \tilde{\gamma}_7 \rightarrow \gamma_1 \]

\[ \Gamma \rightarrow \Gamma \rightarrow \Gamma \rightarrow \tilde{\Gamma} \rightarrow \tilde{\Gamma} \rightarrow \tilde{\Gamma} \rightarrow \pi \rightarrow \pi \rightarrow \pi \]

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\gamma_1 \quad \gamma_2 \quad \gamma_3 \\
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\[ \tilde{\gamma}_1 \rightarrow 1 \rightarrow \tilde{\gamma}_2 \rightarrow \tilde{\gamma}_3 \rightarrow 1 \rightarrow \Gamma \]

\[ \gamma_1 \rightarrow 1 \rightarrow \gamma_2 \rightarrow \gamma_3 \rightarrow \gamma_4 \rightarrow \Gamma \]

\[ \gamma_5 \rightarrow \tilde{\gamma}_5 \rightarrow \tilde{\gamma}_6 \rightarrow \tilde{\gamma}_7 \rightarrow \tilde{\gamma}_1 \]

\[ \pi \]

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Prym varieties of folded $k$-gonal chains of loops
A *Prym tableau* for a given rank $r$ is a square grid of length $r + 1$ where each cell in the grid is filled out with the first $2g - 1$ integers, called *symbols*. 

- The standard condition: every row and column must be strictly increasing.
- The displacement condition: If symbol $n$ repeats in the tableau, then all repeats must be in the same diagonal mod $k$.
- The Prym condition: If symbols $n$ and $2g - n$ both appear in the tableau, they must be in the same diagonal mod $k$. 

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We can index through the tableau by diagonals modulo $k$: For a given symbol $n$ with torsion $k$, the $i$-th diagonal is all cells $(x, y)$ such that $x - y \equiv i \pmod{k}$.
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To go from Prym tableaux to Prym divisor:
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- If $k$ is large enough such that no symbol is allowed to repeat, then the tableaux is called *generic*.
- To go from Prym tableaux to Prym divisor:
  - For symbols less than $g$, measure distance $i$ counterclockwise from the left bridge.
  - For symbols greater than $g$, measure distance $i$ clockwise from the right bridge.
  - For $g$, if $x - y$ is even, then it goes on the top vertex, otherwise it goes on the bottom vertex.
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Prym tableaux

\[
\begin{array}{cccc}
7 & 9 & 10 & 13 \\
5 & 7 & 8 & 12 \\
4 & 6 & 7 & 9 \\
1 & 2 & 5 & 7 \\
\end{array}
\]

-1+1

1
Prym tableaux

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Prym tableaux

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Prym varieties of folded $k$-gonal chains of loops
The correspondence between Prym tableaux and sets of Prym divisors is not in general one-to-one: several different tableaux can yield the same set of divisor classes.
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- A Prym tableaux is *reflective* if, whenever the cell \((x, y)\) contains the symbol \(n\), the cell \((r + 2 - y, r + 2 - x)\) contains the symbol \(2g - n\).
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The dimension is already known in the generic case and when $k$ is even.

Combining our result for $k$ odd with previous results by Len and Ulrisch (2019) in the even and generic cases, we obtain the following theorem:

**Theorem**

$$\dim(V_r) = g - 1 - n,$$

where

$$n = \begin{cases} \frac{(r+1)}{2} & \text{if } r \leq l, \\ \frac{(l+1)}{2} + l \left(r - l \right) & \text{if } r > l, \end{cases}$$

and where $l = \lceil \frac{k}{2} \rceil$. 
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\end{cases}
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and where $l = \left\lceil \frac{k}{2} \right\rceil$.

[1]
Tropical Results

**Theorem**

$V^r$ is pure-dimensional.

**Theorem**

*If $\dim(V^r) > 0$, then $V^r$ is path-connected.*
Enumerating dimension 0

- In the generic case, we can use the well-known hook-length formula.
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Enumerating dimension 0

- In the generic case, we can use the well-known hook-length formula.
- When $k$ is even, we can create a bijection with a lattice path enumeration problem.
- The cardinality is still unknown for $k$ odd.
1-dimensional loci

- Every 1-dimensional locus is a collection of circles wedged together in some way.

\[
\begin{align*}
\text{Theorem} & \quad \text{When } \dim(V_r(\pi)) = 1, \text{ the rank of the first homology of } V_r(\pi) \text{ is:} \\
& \begin{cases}
rf \lambda ((r+1)/2 + 1)^2 + 1 \\
k > 2 \quad 2r - 2 \\
k = 2 \quad 2r - 1 \quad (3r - 2) + 1 \\
k = 4 
\end{cases}
\end{align*}
\]

It is unknown for other values of \( k \).
Every 1-dimensional locus is a collection of circles wedged together in some way.

**Theorem**

When \( \dim(V^r(\pi)) = 1 \), the rank of the first homology of \( V^r(\pi) \) is:

\[
\begin{cases} 
rf^\lambda\left(\frac{(r+1)+1}{2}\right) + 1 & k > 2r - 2 \\
r + 1 & k = 2 \\
2^{r-1}(3r - 2) + 1 & k = 4 
\end{cases}
\]
1-dimensional loci

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**Theorem**

*When \( \dim(V^r(\pi)) = 1 \), the rank of the first homology of \( V^r(\pi) \) is:*

\[
\begin{align*}
&\begin{cases}
rf^\lambda((r+1)+1) + 1 & k > 2r - 2 \\
r + 1 & k = 2 \\
2^{r-1}(3r - 2) + 1 & k = 4
\end{cases}
\]

- It is unknown for other values of \( k \).
Future work

- Continue computing homology groups.
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- Study topological properties of $V^r$ for different covering maps (snake of loops, tree of loops, etc.)
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- Strengthen the connection to Prym divisors on algebraic varieties.
Thank you!