Lower Bounds for the Growth of Erdős-Szekeres Products

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Introduction

Erdős and Szekeres examined the product $\prod_{j=1}^{n} |1 - z^{s_j}|$ in their 1959 paper [1]. The main question they considered was the maximum of this product on the unit circle for given integers $s_1, \ldots, s_n$, which is defined as $M(s_1, \ldots, s_n)$, then considering the minimum of these maximum when taken over all possible choices of n integers; this minimum was defined as $f(n)$. In their paper Erdős and Szekeres obtained the lower bound $f(n) \geq \sqrt{2n}$. Our REU focused on improving this lower bound.

Results so Far

Erdős and Szekeres remark that the lower bound $f(n) \geq \sqrt{2n}$ is nearly trivial. However, it is possible that there is an error in their proof.

They begin their proof by writing $\prod_{j=1}^{n}(1 - z^{s_j}) = \sum_j (z^{a_j}) - \sum_j (z^{b_j})$, which would imply that every coefficient in the polynomial expansion of the product is either -1, 0, or 1. However, this is easy to disprove. Consider, for example, $\prod_{j=1}^{4}(1 - z^j) = z^{10} - z^9 - z^8 + 2z^5 - z^2 - z + 1$, which has a coefficient of 2. We are unable to see if there is some sort of triviality that allows for the consideration of expansions where all coefficients are -1, 0, or 1.

While Erdős and Szekeres do seem to make the mistake above, the method of using the square root of the sum of the squares of the coefficients as a lower bound for $f(n)$ is sound. We use this fact as one of the bases for our program that calculates the sum of the squares of the coefficients of the expanded polynomial.

Our Methods

Our first investigative method was the same method of Erdős and Szekeres, though properly accounting for all coefficients. That is, we tried to minimize the sum of the squares of the coefficients of the product when expanded. In pursuit of this, we developed a computer program that calculates this sum for given a set of integers. We also calculated lists of s_j that produce particularly small maximums. We use $<x_1, \ldots, x_n>$ to denote particular choices for $s_1, \ldots, s_n$. For n up to 7 we have:

n	$<s_1, \ldots, s_n>$	$\lvert < s_1, \ldots, s_n > \rvert \approx$
2	<1, 2>	3.079
3	<1, 2, 3>	4.39
4	<1, 3, 4, 7>	5.693
5	<1, 3, 4, 5, 7>	7.951
6	<1, 2, 3, 4, 5, 7>	7.652
7	<1, 2, 3, 4, 5, 7, 19>	10.643

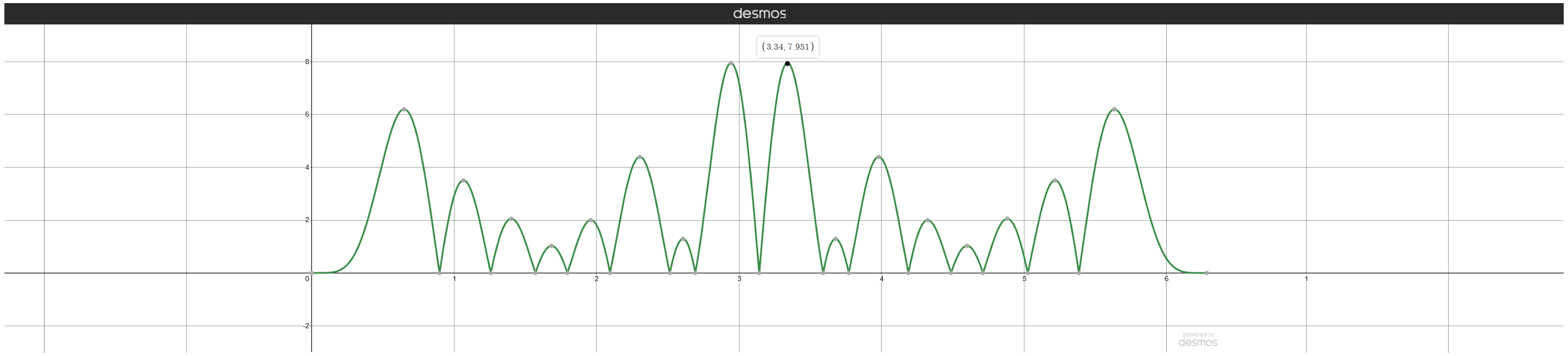
We hope to continue this for larger n and discern a pattern. For the $n = 3$ case we found the exact value of the maximum and the value of z that is occurs. This value is $z = \arccos\left(\frac{-2+\sqrt{10}}{6}\right)$.

We explored avenues to reduce the scope of the possible values of the s_j. Two such avenues were considering if s_1 always had to be 1, and if we show that if the s_j grew too quickly then a small maximum would not be achieved.

(1.1) and (1.2) rely on the s_j to grow slowly and for n to be large. For small n or large s_j, the old bound of $2\sqrt{n}$ is substantially better. (1.1) and (1.2) were found using the Poisson integral representation.

We considered dividing the product by $(1-z)^n$, resulting in a new polynomial $R_n(z)$. The benefit of considering $R_n(z)$ is all its coefficients are positive. The hope was to use the positive coefficients of $R_n(z)$ to learn about the coefficients of the product.

Visualization of the Erdős-Szekeres product



Our Results

We began by noting that the Erdős-Szekeres lower bound of $\sqrt{2n}$ could be slightly improved to $2\sqrt{n}$ using a result by O'Hara and Rodriguez [2]. If $1 \leq L \leq n$, then we have found the following lower bounds.

We also make note that if all the s_j are odd, then the maximum of the corresponding product will be 2^n, the largest possible. Hence the choice of s_j that give rise to $f(n)$ will always have at least one even s_j; this can be seen in our table of particularly small maximums.

$$(1.1) \quad \left\| \prod_{j=1}^{n}(1 - z^{s_j}) \right\|_{L_\infty(|z|=1)} \geq \exp\left(\frac{1}{2e}\frac{L}{(s_1 s_2 \cdots s_L)^{1/L}}\right)$$

$$(1.2) \quad \left\| \prod_{j=1}^{n}(1 - z^{s_j}) \right\|_{L_\infty(|z|=1)} \geq \sqrt{2}\left(\exp\left(\frac{1}{e}\frac{L}{(s_1 s_2 \cdots s_L)^{1/L}} - 1\right)\right)$$

Conclusion/Remarks

The assumptions that produced (1.1) and (1.2) are restrictive on the growth of the s_j and are only potent for large n (about $n \geq 250$). As was noted, this makes the old bound better for large s_j or small n. Further results with less restrictions on growth of the s_j are desirable.

References

[1] P. Erdős, G. Szekeres, *On the product* $\prod_{k=1}^{n}(1 - z^{a_k})$, Publications de L'Institut Mathematique, Academie Serbe des Sciences, 1959, pp. 29-34.

[2] P.J. O' Hara and R.S. Rodriguez, *Some Properties of Self-Inversive Polynomials*, Proc. Amer. Math. Soc., 44(1974), 331-335.