**INTRODUCTION**

Erdős and Szekeres examined the product \( \prod_{j=1}^{n} (1 - z^{s_j}) \) in their 1959 paper [1]. The main question they considered was the maximum of this product on the unit circle for given integers \( s_1, \ldots, s_n \), which is defined as \( M(s_1, \ldots, s_n) \), then considering the minimum of these maximums when taken over all possible choices of \( n \) integers; this minimum was defined as \( f(n) \). In their paper Erdős and Szekeres obtained the lower bound \( f(n) \geq \sqrt{2n} \). Our REU focused on improving this lower bound.

**RESULTS SO FAR**

Erdős and Szekeres remark that the lower bound \( f(n) \geq \sqrt{2n} \) is nearly trivial. However, it is possible that there is an error in their proof. They begin their proof by writing \( \prod_{j=1}^{n} (1 - z^{s_j}) = \sum_n (z^{s_1}) - \sum_n (z^{s_2}), \) which would imply that every coefficient in the polynomial expansion of the product is either \(-1, 0, \) or \(1\). However, this is easy to disprove. Consider, for example, \( \prod_{j=1}^{n} (1 - z) = z^n - z^{n-1} + 2z^{n-2} - z + 1 \), which has a coefficient of 2. We are unable to see if there is some sort of triviality that allows for the consideration of expansions where all coefficients are \(-1, 0, \) or \(1\).

While Erdős and Szekeres do seem to make the mistake above, the method of using the square root of the sum of the squares of the coefficients as a lower bound for \( f(n) \) is sound. We use this fact as one of the bases for our program that calculates the sum of the squares of the coefficients of the expanded polynomial.

**Our Methods**

Our first investigative method was the same method of Erdős and Szekeres, though properly accounting for all coefficients. That is, we tried to minimize the sum of the squares of the coefficients of the product when expanded. In pursuit of this, we developed a computer program that calculates this sum for given a set of integers. We also calculated lists of \( s_j \) that produce particularly small maximums. We use \( <s_1, \ldots, s_n> \) to denote particular choices for \( s_1, \ldots, s_n \). For \( n \) up to 7 we have:

| \( n \) | \( <s_1, \ldots, s_n> \) | \( |<s_1, \ldots, s_n>| \) |
|---|---|---|
| 2 | \( <1, 2> \) | 3.079 |
| 3 | \( <1, 2, 3> \) | 4.39 |
| 4 | \( <1, 3, 4, 7> \) | 5.693 |
| 5 | \( <1, 3, 4, 5, 7> \) | 7.657 |
| 6 | \( <1, 2, 3, 4, 5, 7> \) | 7.657 |
| 7 | \( <1, 2, 3, 4, 5, 7, 10> \) | 10.643 |

We hope to continue this for larger \( n \) and discern a pattern. For the \( n = 3 \) case we found the exact value of the maximum and the value of \( z \) that occurs. This value is \( z = \arccos \left( \frac{-2 + \sqrt{6}}{4} \right) \).

We explored avenues to reduce the scope of the possible values of the \( s_j \). Two such avenues were considering if \( s_j \) always had to be \(1\), and if we show that if the \( s_j \) grew too quickly then a small maximum would not be achieved.

\( (1) \) and \( (2) \) rely on the \( s_j \) to grow slowly and for \( n \) to be large. For small \( n \) or large \( s_j \), the old bound of \( 2\sqrt{n} \) is substantially better. \( (1) \) and \( (2) \) were found using the Poisson integral representation.

We considered dividing the product by \( (1 - z)^n \), resulting in a new polynomial \( R_n(z) \). The benefit of considering \( R_n(z) \) is all its coefficients are positive. The hope was to use the positive coefficients of \( R_n(z) \) to learn about the coefficients of the product.

**Visualization of the Erdős-Szekeres Product**

**Our Results**

We began noting that the Erdős-Szekeres lower bound of \( \sqrt{2n} \) could be slightly improved to \( 2\sqrt{n} \) using a result by O’Hara and Rodriguez [2]. If \( 1 \leq L \leq n \), then we have found the following lower bounds.

We also note that if all the \( s_j \) are odd, then the maximum of the corresponding product will be \( 2^n \), the largest possible. Hence the choice of \( s_j \) that give rise to \( f(n) \) will always have at least one even \( s_j \); this can be seen in our table of particularly small maximums.

\[
\begin{align*}
(1.1) \quad \left\| \prod_{j=1}^{n} (1 - z^{s_j}) \right\|_{L_n[|z|=1]} &\geq \exp \left( \frac{1}{26} \frac{L}{(s_1s_2 \cdots s_L)^{1/7}} \right) \\
(1.2) \quad \left\| \prod_{j=1}^{n} (1 - z^{s_j}) \right\|_{L_n[|z|=1]} &\geq \sqrt{2} \left( \exp \left( \frac{1}{26} \frac{L}{(s_1s_2 \cdots s_L)^{1/7}} \right) - 1 \right)
\end{align*}
\]

**Conclusion/Remarks**

The assumptions that produced \( (1) \) and \( (2) \) are restrictive on the growth of the \( s_j \) and are only potent for large \( n \) (about \( n \geq 250 \)). As was noted, this makes the old bound better for large \( s_j \) or small \( n \). Further results with less restrictions on growth of the \( s_j \) are desirable.

**References**

1. P. Erdős, G. Szekeres, *On the product* \( \prod_{k=1}^{n} (1 - z^{n_k}) \), Publications de L’Institut Mathematique, Academie Serbe des Sciences, 1959, pp. 29-34.