Approximating Banded Toeplitz Matrices by Circulants

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Background

Toeplitz Matrix : a Toeplitz matrix is a matrix where the same entries are down the diagonals. Meaning its is an n x n matrix $T_n = [t_{k,j};\ k,j = 0,1,\ldots,n-1]$ where $t_{k,j} = t_{k-j}$, i.e., a matrix of the form

$$T_n = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & & \\ t_2 & t_1 & t_0 & & \vdots \\ \vdots & & & \ddots & \\ t_{n-1} & & & \cdots & t_0 \end{bmatrix}$$

They arise in the solution t differential and integral equations, spline functions, and process and methods in physics, mathematics, statistics, and signal processing

Circulant Matrix : is a common special case of Toeplitz matrices. It is a matrix where every row of the matrix is a right cyclic shift of the row above it, so that $t_k = t_{-(n-k)} = t_{k-n}$. Becoming:

$$C_n = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(n-1)} \\ t_{-(n-1)} & t_0 & t_{-1} & & \\ t_{-(n-2)} & t_{-(n-1)} & t_0 & & \vdots \\ \vdots & & & \ddots & \\ t_{-1} & t_{-2} & & \cdots & t_0 \end{bmatrix}$$

They arise in applications involving the discrete Fourier transform (DFT) and the study of cyclic codes for error correction.
They are remarkable in that we can explicitly find their eigenvalues, eigenvectors, determinants, etc.

Banded Toeplitz Matrix: it is a Toeplitz matrix possessing a finite number of diagonals with nonzero entries and zeros everywhere else.
They sometimes make easier the study of Toeplitz matrices. For example, the Toeplitz matrix below would be considered a Toeplitz matrix with 2m+1 number of bands.

$$T_n = \begin{bmatrix} t_0 & t_{-1} & \cdots & t_{-m} & & & & & & \\ t_1 & t_0 & & & & & & & & \\ \vdots & & & & & & 0 & & & \\ & & \ddots & & & \ddots & & & & \\ t_m & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & t_m & \cdots & t_1 & t_0 & t_{-1} & \cdots & t_{-m} & \\ & & & & & & & & \ddots & \\ & & & & \ddots & & \ddots & & & t_{-m} \\ & & & & & & & & & \vdots \\ & 0 & & & & & & & t_0 & t_{-1} \\ & & & & & & t_m & \cdots & t_1 & t_0 \end{bmatrix}.$$

Turning a Banded Toeplitz matrix into a Circulant Matrix

Because we know the formulas for eigenvalues and determinants of a cirulant matrix but not of a Toeplitz Matrix. Banded Toeplitz matrices with not too many bands can be turned into circulants by adding the appropriate entries in the top right and bottom left corners of the matrix.

$$C_n = \begin{bmatrix} t_0 & t_{-1} & \cdots & t_{-m} & & & & t_m & \cdots & t_1 \\ t_1 & & & & & & & & \ddots & \vdots \\ & & & & & & & & & t_m \\ \vdots & & & & \ddots & & & & & \\ t_m & & & & & & & 0 & & \\ & \ddots & & & & & & & & \\ & & t_m & \cdots & t_1 & t_0 & t_{-1} & \cdots & t_{-m} & \\ & & & \ddots & & & & & \ddots & \\ & & & 0 & & & & & & t_{-m} \\ t_{-m} & & & & & & & & & \\ \vdots & \ddots & & & & & & & & \vdots \\ & & & & & & & & t_0 & t_{-1} \\ t_{-1} & \cdots & t_{-m} & & & & t_m & \cdots & t_1 & t_0 \end{bmatrix}$$

Motivation

Eigenvalues and determinants of Toeplitz matrices are heavily used in physics and statistics, especially how they behave as matrix size grows to infinity.

Idea

$$\text{Cn- Tn= Dn=} \begin{bmatrix} 0 & 0 & 0 & t_{-m} & t_{-1} \\ 0 & 0 & 0 & 0 & t_{-m} \\ 0 & 0 & 0 & 0 & 0 \\ t_m & 0 & 0 & 0 & 0 \\ t_1 & t_m & 0 & 0 & 0 \end{bmatrix}$$

To estimate the powers of Cn-Tn and use
Tr(A) = sum of eigenvalues
$Tr(A^r)$= sum of r^{th} power of eigenvalues where $r \geq 1$
So hopefully $Tr(Tn^r)$ where r=1,2,3… is close to $Tr(Cn^r)$ and can then at least on average compare eigenvalues of Tn and Cn in some ways

Results

The results are conveying the idea that the matrices are not dependent on matrix size but on m and r.

Assume:

- the matrices have a fixed m and Tn has 2m+1 bands
- All entries of $|t\pm j| \leq 1$
- Tn has eigenvalues $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$
- Cn has eigenvalues $\gamma_1, \gamma_2, \gamma_3 \ldots \gamma_n$

Result 1:
If r is a non-negative integer,

$$\lim_{n\to\infty} \frac{1}{n}\left(\sum_{j=1}^{n} \lambda_j^r - \sum_{j=1}^{n} \gamma_j^r\right) = 0$$

Result 2:
Assume $f:\mathbb{R} \to \mathbb{R}$ is continuous and equals zero outside some bounded interval

$$\lim_{n\to\infty} \frac{1}{n}\left(\sum_{j=1}^{n} f(\lambda_j) - \sum_{j=1}^{n} f(\gamma_j)\right) = 0$$

For future work, we will consider when m is not fixed.

References

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