Ramsey Numbers in Grid Graphs

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Setup

Ramsey theory focuses on finding specific small structures within arbitrary large structures.

We focused on Ramsey problems within a structure called a grid graph. A *grid graph* $G_{M\times N}$ is a graph whose vertices are points on the integer lattice with M columns and N rows, and two vertices are adjacent if they share one coordinate.

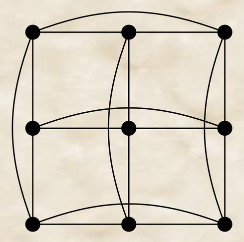


Figure 1. $G_{3\times 3}$

Let H be a graph where all edges are labeled vertical and horizontal.

Definition ($\mathrm{gr}(H, K_k)$). We let the grid Ramsey number of H against K_k, denoted $\mathrm{gr}(H, K_k)$, be that the smallest N such that every red-blue edge-coloring of $G_{N\times N}$ contains either a red copy of H or a blue clique of size k.

This project focused on finding the asymptotics of $\mathrm{gr}(H, K_k)$ as we fix H and vary k as a parameter. We are mostly interested in whether $\mathrm{gr}(H, K_k) = k^{O(1)}$ or not.

Theorem (Conlon, Fox, He, Mubayi, Suk, Verstraëte). If H is a rectangle (a 4-cycle with alternating horizontal and vertical edges) then $\mathrm{gr}(H, K_k) > 2^{c(\log k)^2}$ for some c.

Why Study Ramsey theory on Grid Graphs?

A 3-uniform hypergraph $G = (V, E)$ is a set of vertices V and a set of (hyper)edges $E \subseteq \binom{V}{3}$ where each (hyper)edge is a set of three vertices. If a 3-uniform hypergraph $G = (V, E)$ has vertices that can be bipartitioned $V = X \sqcup Y$ such that for any edge $e \in E$, there exists $x \in X$ and $y \in Y$ such that $x, y \in e$, we say G has "property B."

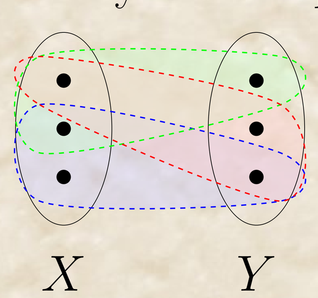


Figure 2. Example of a 3-uniform hypergraph with property B

For a 3-uniform hypergraph $G = (V, E)$ with property B and relevant bipartition $V = X \sqcup Y$, we can create a grid subgraph H on $|X|$ columns and $|Y|$ rows by the rule that $(x_1, y) \sim (x_2, y)$ if $\{x_1, x_2, y\} \in E$ and $(x, y_1) \sim (x, y_2)$ if $\{x, y_1, y_2\} \in E$. Then the Ramsey number of G versus a k-star is bounded above by $2\,\mathrm{gr}(H, K_k)$.

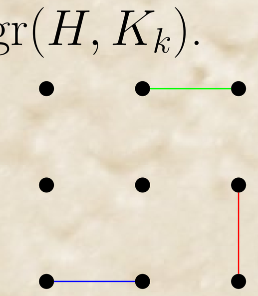


Figure 3. Conversion of Above Example to Grid Graphs

Row and Column Duplication

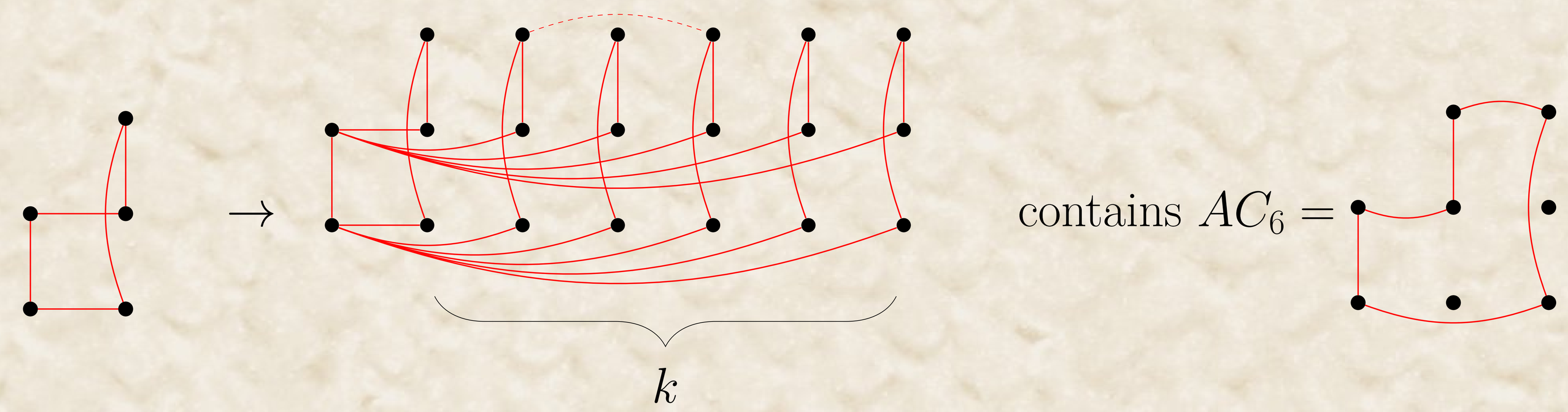
In characterizing polynomial versus superpolynomial growth of grid Ramsey numbers, it's convenient to come up with general techniques to build graphs iteratively. A standard supersaturation argument shows that if a graph H has sufficient density in any blue K_k-free grid graph of a given polynomial order, then we can find many copies of H in a blue K_k-free grid graph of some higher polynomial order. Some of these copies are bound to overlap, giving us a natural way to build up larger graphs.

Theorem 1. Let H' be a grid graph constructable from applying a constant number of steps of the below method starting from a single point. Then for all subgraphs H of H', $\mathrm{gr}(H, K_k) = k^{O(1)}$.

The Method

- Pick a column (or row) x in H.
- Create duplicate column x', and for each edge that intersects column x, create a copy of the edge intersecting x'.
- Draw (any) one additional horizontal edge between columns x and x'. This extra edge comes for free in the proof. We note that in the hypergraph setting, this row/column duplication corresponds to "twinning" a vertex and adding an extra edge. Thus, we can repeat this step for any vertex blowup.

Example of Column Duplication to get AC_6



Generalized Subdivision

If we narrow our focus, the method above lets us replace a given edge of H by an arbitrary graph lying entirely within that row or column. We call this a *generalized subdivision* of an edge. In particular, in n duplication steps as above, we can replace a given edge with a copy of K_n.

Corollary. Let $\mathrm{gr}(H, K_k) = k^{O(1)}$. If H' is a grid graph constructed from H by the *generalized subdivision* of an edge. Then $\mathrm{gr}(H', K_k) = k^{O(1)}$.

Together, these techniques immediately yield a large class of graphs with polynomial grid Ramsey number, though they do not always produce the best bound. The following column highlights some of our more granular results.

Trees

Theorem 2. Let T be a tree. Then $\mathrm{gr}(T, K_k) = O(k)$.

Alternating Cycles

Definition. An Alternating cycle is a cycle with edges alternating between row-edges and column edges. We denote an Alternating Cycle with $2d$ edges by AC_{2d}.

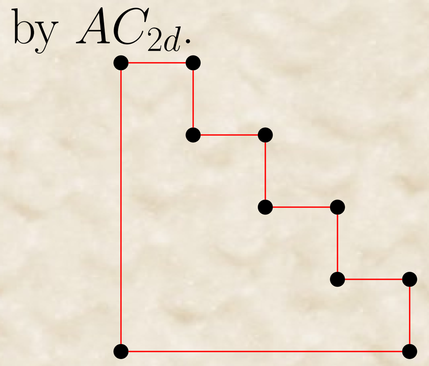


Figure 4. AC_{10}

Theorem 3. For $d \geq 3$, $\mathrm{gr}(AC_{2d}, K_k) = k^{O(d)}$.

Bounds on AC_6

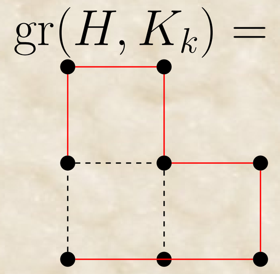
We have the following lower and upper bounds on $\mathrm{gr}(AC_6, K_k)$.

Theorem 4. There exist constants c_1, c_2 such that $c_1 k^2/\log(k) \leq \mathrm{gr}(AC_6, K_k) \leq c_2 k^3$.

Further Directions

Question 1. Improve bounds on $\mathrm{gr}(H, K_k)$ for certain values of H, especially $H = AC_6$.

Question 2. Generalize to forcing some vertices of H to align vertically/horizontally. Is $\mathrm{gr}(H, K_k) = k^{O(1)}$ true?



Question 3. Can we classify graphs H for which $\mathrm{gr}(H, K_k)$ is polynomial in k contingent on the Multicolor Erdős-Hajnal Conjecture?

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