Mapping Class Groups, Covering Spaces, and Symplectic Matrices

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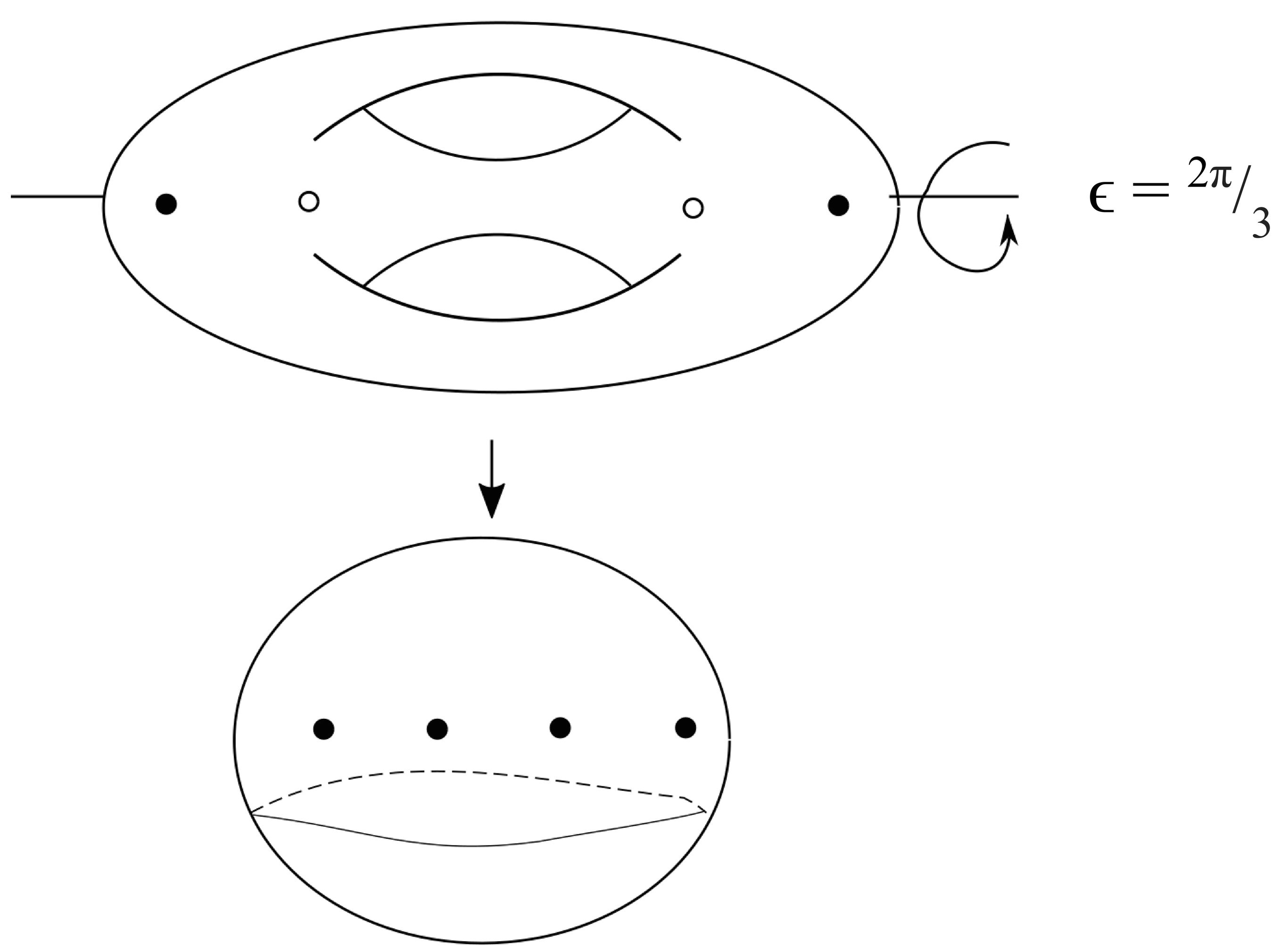
Mapping Class Group

$\mathrm{Mod}(S_g) = \mathrm{Homeo}(S_g)/\text{homotopy}$

$\mathbf{\Phi}: \mathrm{Mod}(S_g) \to \mathrm{Sp}(2g, \mathbb{Z})$ induced by the action on $H_1(S_g)$

Rotation ϵ



$$\mathbf{\Phi}(\epsilon) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Symmetric Mapping Class Group

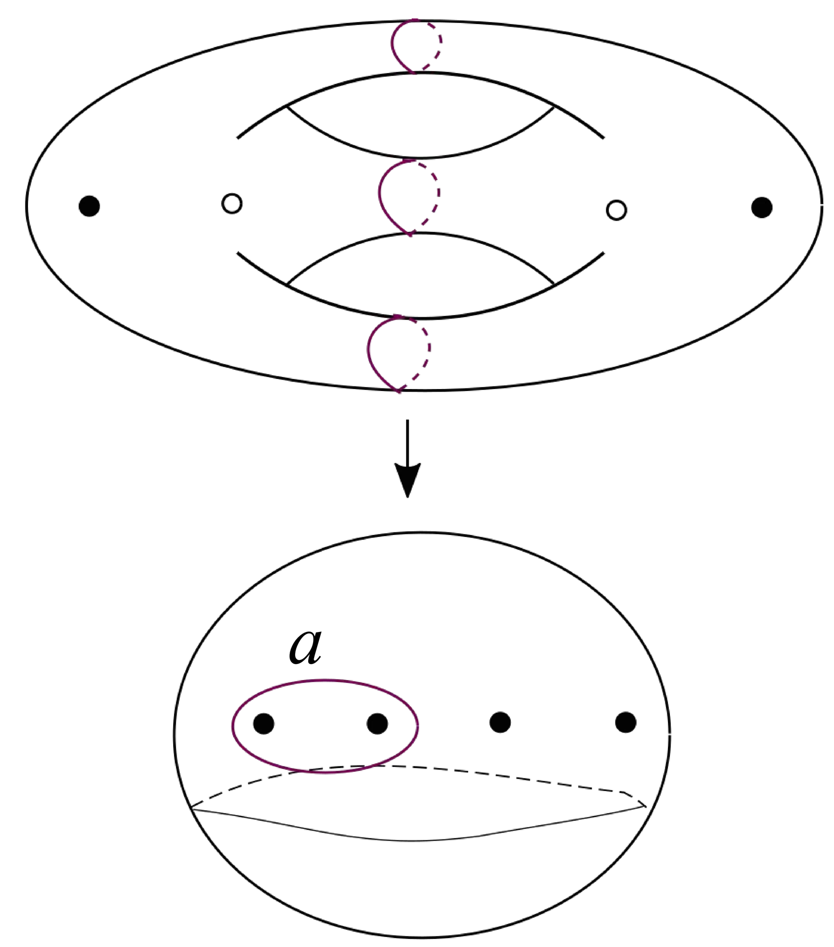
$\mathrm{SMod}(S_2)$ is the homotopy classes of fiber-preserving homeomorphisms of S_2.

Question

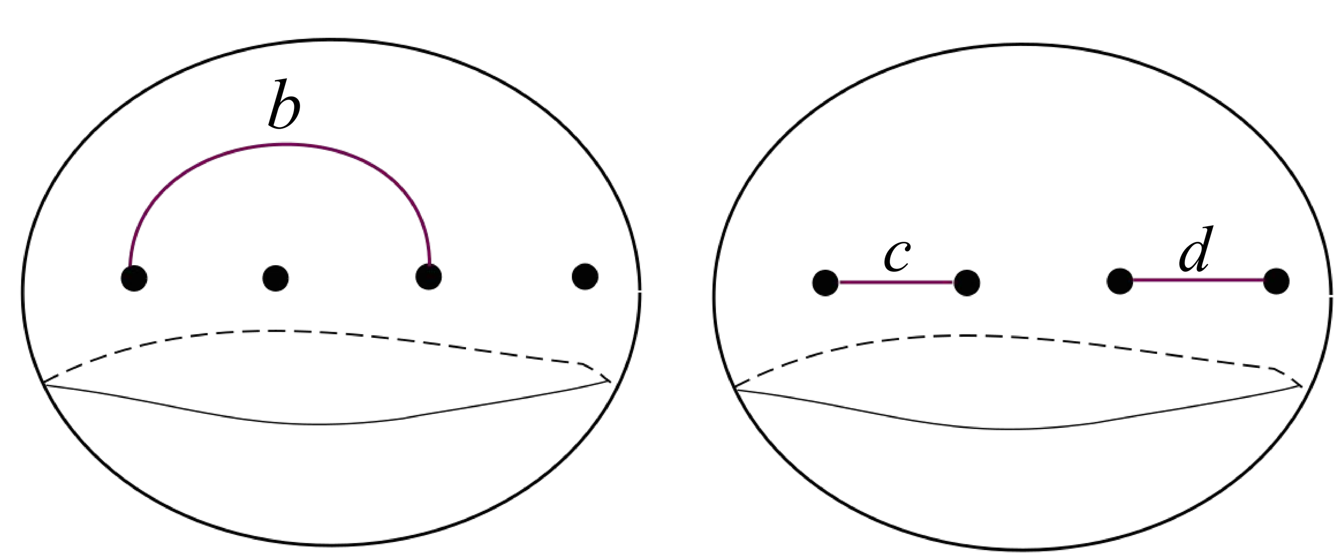
What is the image of $\mathrm{SMod}(S_2)$ in $\mathrm{Sp}(4, \mathbb{Z})$?

Note: $\mathbf{\Phi}(\mathrm{SMod}(S_2)) \subseteq N_{\mathrm{Sp}(4,\mathbb{Z})}(\langle\mathbf{\Phi}(\epsilon)\rangle)$

Elements of $\mathrm{SMod}(S_2)$



A Dehn twist about a in $S_{0,4}$ lifts to a composition of Dehn twists about three curves in S_2, denoted T_A.



The half twist about b lifts to B. The composition of half twists about c and d lifts to C.

Strategy

Step 1: Calculate $N_{\mathrm{Sp}(4,\mathbb{Z})}(\langle\mathbf{\Phi}(\epsilon)\rangle)$ in MATLAB

Output: 12 infinite families of matrices

Step 2: Find $\mathbf{\Phi}(g_i)$ for g_i generators of $\mathrm{SMod}(S_2)$. Ghaswala-Winarski give the generators as lifts of homeomorphisms in $S_{0,4}$.

Step 3: Find a product of matrices in $\mathbf{\Phi}(\mathrm{SMod}(S_2))$ for each element of the normalizer.

Example:

$$M = \begin{bmatrix} 2x & 1 & -x & 0 \\ -1 & 0 & 0 & 0 \\ x & 0 & -2x & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$M = \mathbf{\Phi}(B \circ T_A^{\,x} \circ C)$$

Theorem (Davis-Stordy-Zhou)

$$\mathbf{\Phi}(\mathrm{SMod}(S_2)) = N_{\mathrm{Sp}(4,\mathbb{Z})}(\langle\mathbf{\Phi}(\epsilon)\rangle)$$

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