

Abstract

Symmetric edge polytopes are a type of polytope formed from a finite, simple graph. A common question about these kinds of polytopes is how their facets can be classified, and how certain properties of a graph will determine what its symmetric edge polytope will look like. Previous work on symmetric edge polytopes gives us theorems and methods for counting facets using certain functions and maximal bipartite subgraphs. With these tools, we can make progress towards determining which graphs on a given number of vertices and edges will maximize the number of facets in their symmetric edge polytopes.

Background and Preliminaries

Definition: A **polytope** $P \subseteq \mathbb{R}^d$ is the convex hull of finitely many points in $\mathbb{R}^d$.

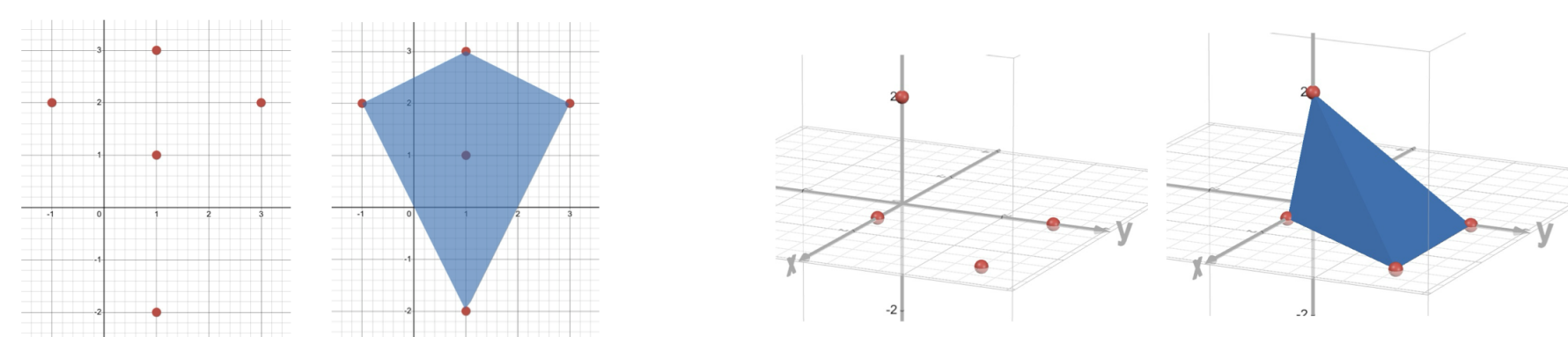


Figure 1. Examples of Point Sets and Their Corresponding Polytopes

Definition: A hyperplane $H = \{x \in \mathbb{R}^d \mid \langle a, x \rangle = b\}$ is a **supporting hyperplane** of P if all points of P lie on one side of H and $H \cap P \neq \emptyset$.

Definition: $F \subseteq \mathbb{R}^d$ is a **face** of P if $F = P \cap H$ for some supporting hyperplane H . A **facet** is a face of dimension one less than the polytope itself. (In figure 1, the facets of the left polytope are the bounding lines and the facets of the right polytope are the bounding triangles.)

Definition: Given a finite, simple graph $G = (V, E)$, we define the **symmetric edge polytope** to be $P_G = \text{conv}(\pm(e_i - e_j) \mid (i, j) \in E) \subset \mathbb{R}^V$, where $e_v \in \mathbb{R}^V$ is the unit vector indexed by the vertex v of G .

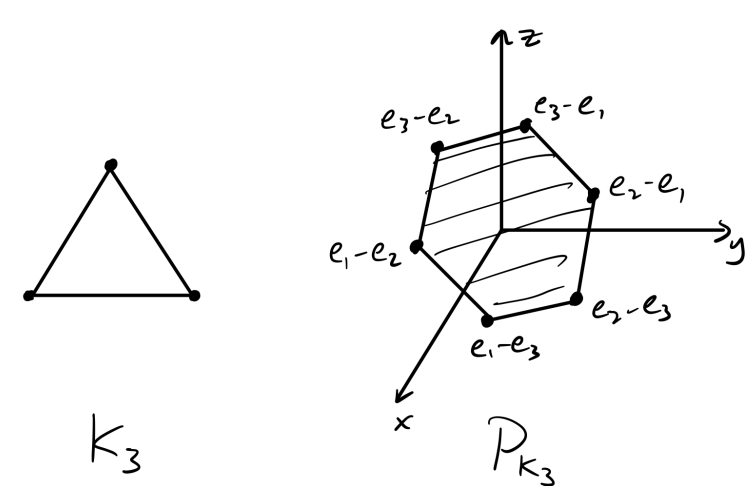


Figure 2. The graph K_3 along with its symmetric edge polytope P_{K_3}

Classifying Facets

Definition: Let $G = (V, E)$ be a finite simple connected graph, let P_G be its symmetric edge polytope, and let $f : V \rightarrow \mathbb{Z}$ be a function. We say f is **facet-defining** if the hyperplane $H_f = \{x \in \mathbb{Z}^V : \sum_{v \in V} f(v)x_v = 1\}$ supports P_G and $P_G \cap H_f$ is a facet of P_G .

Theorem (Higashitani, Jochemko, Michalek, 2019): Let $G = (V, E)$ be a finite simple connected graph. Then $f : V \rightarrow \mathbb{Z}$ is facet-defining if and only if both of the following hold:

1. For any edge $e = (u, v)$ we have $|f(u) - f(v)| \leq 1$
2. The subset of edges $E_f = \{e = (u, v) \in E : |f(u) - f(v)| = 1\}$ forms a spanning connected subgraph of G .

Lemma (Chen, Davis, Korchevskaia, 2023): Let G be a connected graph. A subgraph H of G is a facet subgraph of G if and only if it is a maximal connected spanning bipartite subgraph of G .

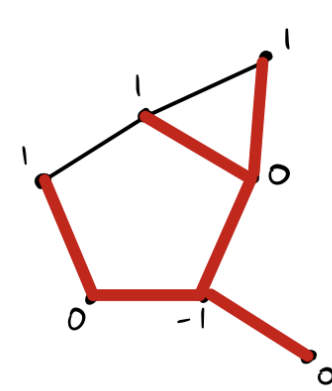


Figure 3. A graph with a valid facet-defining function (labels on vertices) and facet subgraph highlighted in red. This allows us to determine the number of facets the symmetric edge polytope of a graph G will have by counting the number of maximal spanning connected bipartite subgraphs of G and multiplying by the number of valid facet-defining functions f of vertices.

Symmetric Edge Polytopes of Graphs Containing Cycles

Cycles may have a simple structure but they play an essential role in determining maximal spanning connected bipartite subgraphs of a given graph. A bipartite graph contains no odd cycles, therefore, to determine the maximal spanning connected bipartite subgraphs of a graph, we count the number of ways to break all odd cycles in the graph.

Cycles Sharing Vertices

Definition: For graphs G and H , let $G \vee H$ denote a graph obtained by identifying a vertex in G with a vertex in H . We call $G \vee H$ a **wedge**.

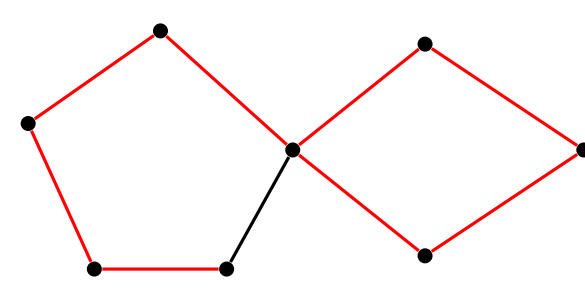


Figure 4. $C_5 \vee C_4$ has 5 maximal spanning connected bipartite subgraphs.

Proposition (C. 2026): Let C_n denote a cycle with n edges and C_m denote a cycle with m edges. The number of maximal spanning connected bipartite subgraphs of $C_n \vee C_m$ is $n \cdot m$ if m and n are odd, n if n is odd, or 1 if both n and m are even.

Cycles Sharing Edges

Let C_n denote a cycle with n edges and C_m denote a cycle with m edges. Let $C_n \cup C_m$ denote C_n and C_m joined at an edge.

Proposition (C. 2026): Given two cycles C_n and C_m we can determine the number of maximal connected spanning bipartite subgraphs of $C_n \cup C_m$ as follows:

1. If n and m are odd, there are $(n-1)(m-1)+1$.
2. If n is odd and m is even, there are $n+m-2$.
3. If n and m are even, there is 1.

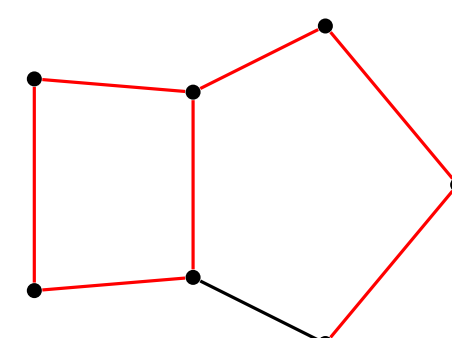


Figure 5. $C_4 \cup C_5$ has 7 maximal spanning connected bipartite subgraphs.

Proposition (C. 2026): Given three odd cycles $C_{n_1}, C_{n_2}, C_{n_3}$, n_1, n_2, n_3 odd, the number of maximal connected spanning bipartite subgraphs of $C_{n_1} \cup C_{n_2} \cup C_{n_3}$ is $(n_1-1)((n_2-2)(n_3-1)+1) + (n_2-2) + (n_3-1)$.

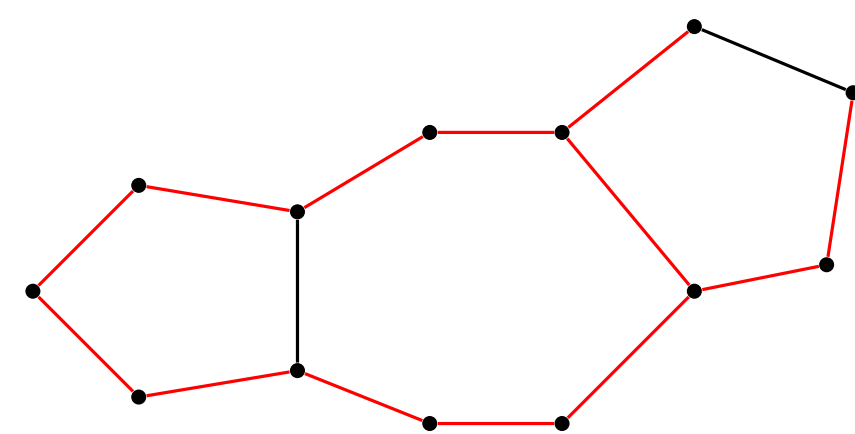


Figure 6. $C_5 \cup C_7 \cup C_5$ has 93 maximal spanning connected bipartite subgraphs.

Proposition (C. 2026): Given three odd cycles $C_{n_1}, C_{n_2}, C_{n_3}$ such that each cycle shares exactly one edge with each of the other two cycles, the total number of maximal connected spanning bipartite subgraphs is $(n_1-2)(n_2-2)(n_3-2) + 2((n_1-2) + (n_2-2) + (n_3-2))$.

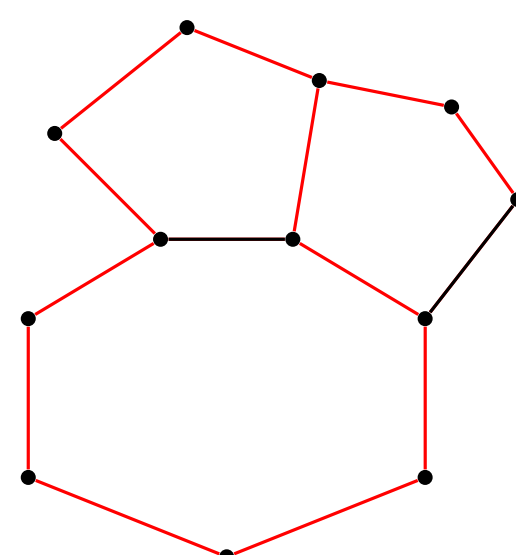


Figure 7. This graph containing cycles C_7, C_5, C_5 has 67 maximal spanning connected bipartite subgraphs.

Conjectures and Results

Using these formulas, we can analyze certain types of graphs containing these structures. By counting their maximal connected spanning bipartite subgraphs, we can determine which graphs with a given number of vertices and edges will have the most facets in their symmetric edge polytopes.

Windmill Graphs

Definition: A **full windmill** is a wedge of $(n-1)/2$ triangles at a single vertex. Denote by $WM(n)$ the full windmill on n vertices.

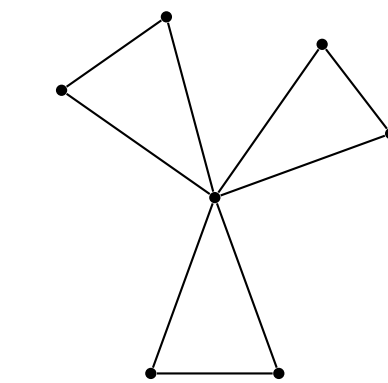


Figure 8. $WM(7)$

Conjecture (Braun, Bruegge, 2023): Among graphs with n vertices and $3(n-1)/2$ edges (for odd n), $WM(n)$ is a facet-maximizer.

Lemma (C. 2026): For graphs with 7 vertices and 9 edges, $WM(7)$ is a facet-maximizer.

Graphs with n Vertices, $n+k$ Edges

Theorem (Braun, Bruegge, 2023): For graphs with n vertices and n edges, the facet-maximizing graph is C_n for odd n and $C_{n-1} \vee e$ for even n , where e is an edge.

Conjecture (C. 2026): For graphs with n vertices and $n+2$ edges, the facet-maximizing graph is a wedge of three odd cycles when n is odd, and two odd cycles sharing an edge wedged with an odd cycle when n is even.

This conjecture was verified computationally using Sage Math for $n \leq 11$.

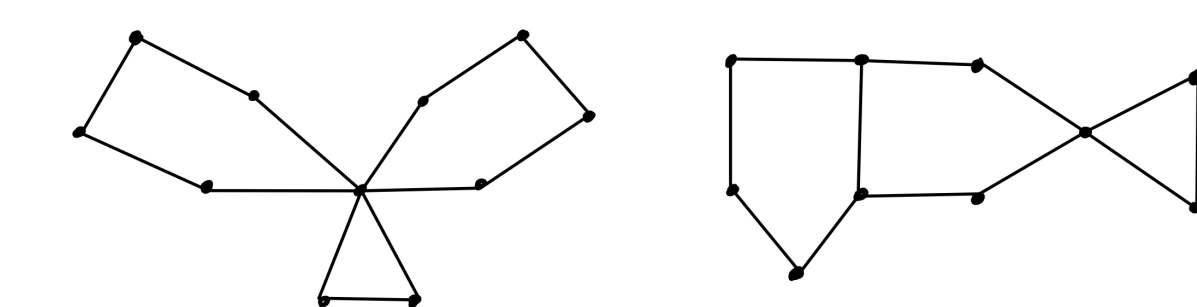


Figure 9. Facet-maximizing graphs for $n = 11$ and $n = 10$

Lemma (C. 2026): Suppose we have a wedge of three cycles on n vertices and $n+2$ edges, where n is odd and $n \geq 7$. The facet-maximizing graph can be characterized below:

1. If $n+2 = 3k$, the facet-maximizing graph is $C_k \vee C_k \vee C_k$.
2. If $n+2 = 3k+1$, the facet-maximizing graph is $C_{k+1} \vee C_{k+1} \vee C_{k-1}$.
3. If $n+2 = 3k+2$, the facet-maximizing graph is $C_{k+2} \vee C_k \vee C_k$.

Lemma (C. 2026): Let n be odd, $n \geq 7$. Suppose G is the facet-maximizing graph on n vertices and $n+2$ edges, and H is the facet-maximizing graph on $n+2$ vertices and $n+4$ edges, where G and H are wedges of three cycles. Let k_1, k_2, k_3 denote the cycle lengths of H , where $k_1 \geq k_2 \geq k_3$. Then the facet-maximizing graph on $n+1$ vertices, $n+3$ edges is $C_{k_1} \cup C_{k_2} \vee C_{k_3}$.

Conjecture (C.2026): Given n vertices, $n+k$ edges, n odd and $n \geq 2k+1$, the facet-maximizing graph is a wedge of $n+1$ odd cycles.

Acknowledgments and References

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