Graphs of Groups with Legs and their Zeta Functions

Noah Parker

Georgia Institute of Technology, School of Mathematics

Graphs with Legs

A **graph with legs** X consists of

- A set of **vertices** $V(X)$,
- A set of **half-edges** $H(X)$,
- A **root map** $r : H(X) \to V(X)$,
- An **involution** $h \to \overline{h}$ on $H(X)$.

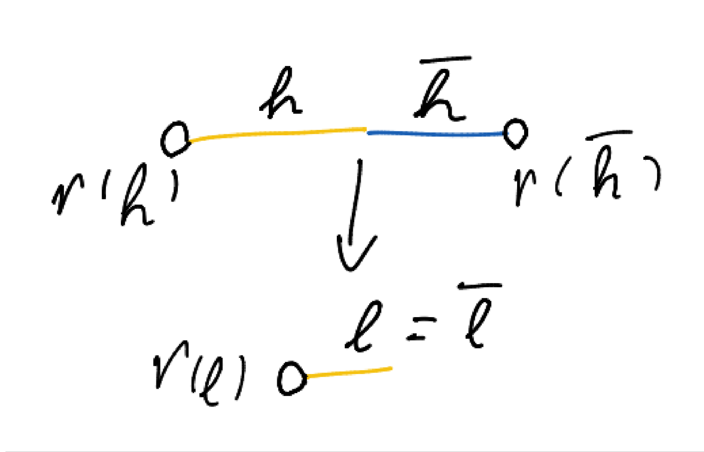

Figure 1. A graph with legs and its quotient by $\mathbb{Z}/2\mathbb{Z}$.

Motivation

Let G be a finite group acting on a finite graph X. We take X/G to be the graph with $V(X/G) = V(X)/G$ and $H(X/G) = H(X)/G$. However, if G flips an edge of X, then X/G is not a graph, but a **graph with legs**.

Even when X/G is a graph without legs, we can find X and G such that

$$|\operatorname{Jac}(X/G)| \nmid |\operatorname{Jac}(X)|$$

and

$$\zeta(u, X/G) \nmid \zeta(u, X).$$

We define a **quotient graph of groups with legs** so these divisibility results will hold. This appraoch was used in Meyer and Zakharov (2023) to show that

$$|\operatorname{Jac}(X//G)| \mid |\operatorname{Jac}(X)|$$

for an arbitrary group action G on X. We begin the work necessary to extend the results of Zakharov (2021) to arbitrary group actions as well, generalizing results from Bass (1992) and Bass (1993) to the graphs with legs case along the way.

Graphs of Groups with Legs

A **graph of groups with legs** $\mathbb{X} = (X/G, \{\mathcal{X}_v\}, \{\mathcal{X}_h, i_h\})$ consists of

- A graph with legs X.
- A group $\mathcal{X}_v$ for each vertex $v \in V(X)$.
- A subgroup $\mathcal{X}_h \subset \mathcal{X}_{r(h)}$ for each half-edge $h \in H(X)$.
- An isomorphism $i_h : \mathcal{X}_h \to \mathcal{X}_{\overline{h}}$ for each half-edge $h \in H(X)$, such that each $i_{\overline{h}} \circ i_h : \mathcal{X}_h \to \mathcal{X}_h$ is an inner automorphism.

Quotient Graphs of Groups with Legs

If a finite group G acts on a finite graph X, we can form the **quotient graph of groups with legs**

$$X//G = (X/G, \{\mathcal{X}_v\}, \{\mathcal{X}_h, i_h\})$$

by setting

$$\mathcal{X}_v = \operatorname{Stab}_G(\widetilde{v}),$$
$$\mathcal{X}_h = \operatorname{Stab}_G(\widetilde{h})$$

and i_h a suitable inner automorphism. This gives us a graph of groups structure on X/G.

An Example of a Quotient Graph of Groups with Legs

Here we see the quotient graph K_4/S_3, where S_3 acts transitively on $\{\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3\} \subset V(K_4)$.

The quotient graph of groups $K_4//S_3$ has

$$\mathcal{X}_u = \operatorname{Stab}_{S_3}(\widetilde{u}) = S_3$$

$$\mathcal{X}_h = \operatorname{Stab}_{S_3}(\widetilde{h}) = \langle (2\,3) \rangle$$

$$\mathcal{X}_{\overline{h}} = \operatorname{Stab}_{S_3}(\widetilde{\overline{h}}) = \langle (2\,3) \rangle$$

$$\mathcal{X}_v = \operatorname{Stab}_{S_3}(\widetilde{v}_1) = \langle (2\,3) \rangle$$

$$\mathcal{X}_\ell = \operatorname{Stab}_{S_3}(\widetilde{\ell}) = 1.$$

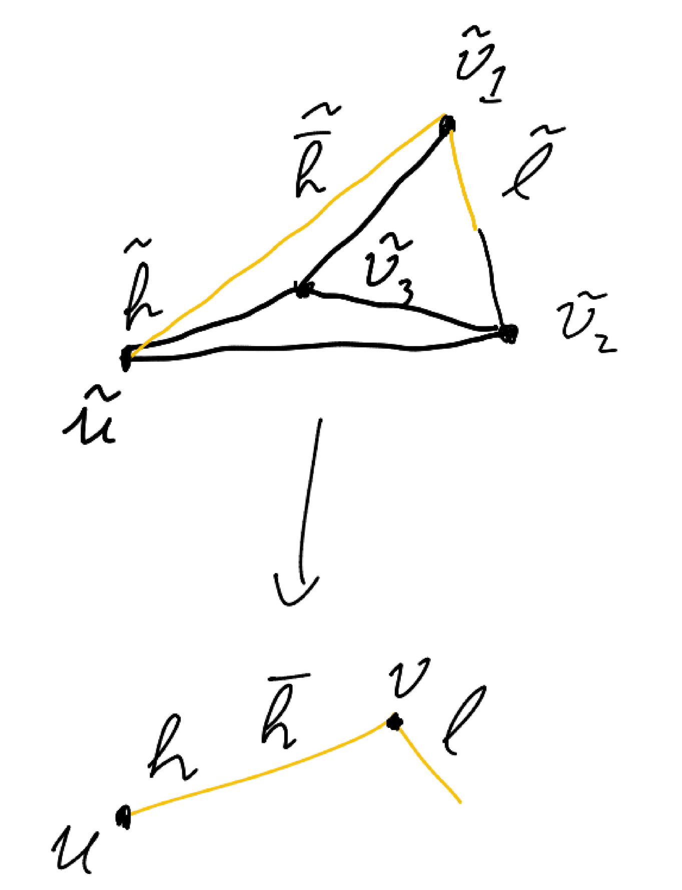

Figure 2. K_4 lying over its quotient graph K_4/S_3

Cycles in Graphs of Groups with Legs

A **cycle** in a graph of groups $\mathbb{X} = (X, \{\mathcal{X}_v\}, \{\mathcal{X}_h, i_h\})$ is a sequence

$$\gamma = (g_1,\ h_1,\ g_1, \dots,\ g_n,\ h_n)$$

of length $\ell(n) := n$, where $g_j \in \mathcal{X}_{r(h_j)}$ for all $1 \leqslant j \leqslant n$, and $(h_1, \dots, h_n)$ is a cycle in the base graph X.

We say that γ is **cycle reduced** if $h_j = \overline{h}_{j-1}$ implies $g_j \notin \mathcal{X}_{h_j}$ for $1 < j \leqslant n$, and $h_1 = \overline{h_n}$ implies $g_1 \notin \mathcal{X}_{h_1}$.

We say that γ is **primitive** if there is not another cycle η such that $\gamma = \eta^k$, the concatenation of $k > 1$ copies of η.

Fix sets of coset representatives S_h of $\mathcal{X}_{r(h)}/\mathcal{X}_h$ containing 1 for all $h \in H(X)$.
We say that $\gamma = (g_1, h_1, \dots, g_n, h_n)$ is S**-reduced** if it is cycle reduced, and $g_j \in S_{h_j}$ for all $1 \leqslant j \leqslant n$.

We call

$$[\gamma] = \{(g_1,\ h_1, \dots,\ g_n,\ h_n),$$
$$(g_2,\ h_2, \dots,\ g_1,\ h_1),\ \dots\}$$

the **rotation class** of γ.

A rotation class of primitive, S-reduced cycles is called a **prime** of $\mathbb{X}$.

Examples of Cycles

Recall the above example of $K_4//S_3$. The cycles

$$\gamma = ((1\,2),\ h,\ 1,\ \ell,\ 1,\ \overline{h}),$$

$$\eta = ((2\,3),\ \ell,\ (2\,3),\ \ell)$$

are both reduced.

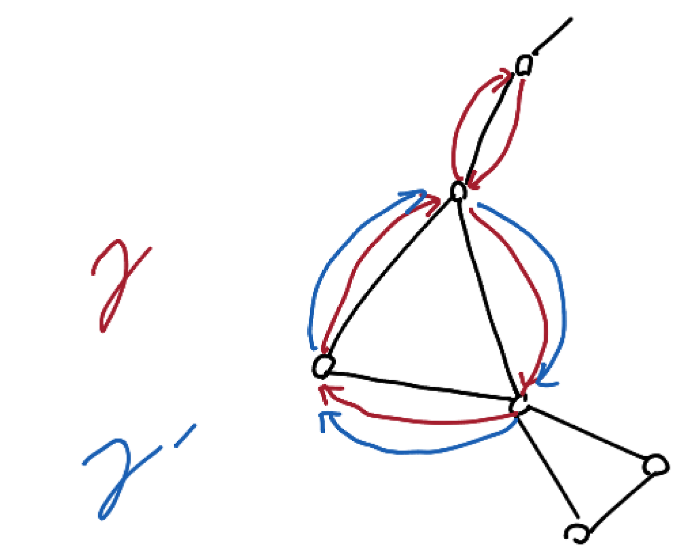

Figure 3. An unreduced path γ and reduced path γ' in a trivial graph of groups with legs.

Ihara Zeta Function of Graphs of Groups with Legs

The **Ihara zeta function** $\zeta(u, \mathbb{X})$ of a graph of groups with legs $\mathbb{X}$ is

$$\zeta(u, \mathbb{X}) = \prod_{\mathfrak{p}} \left(1 - u^{\ell(\mathfrak{p})}\right)^{-1},$$

where the product is taken over all primes $\mathfrak{p}$ of $\mathbb{X}$.

Result: Two-term determinant formula

Let $\mathbb{X}$ be a graph of groups with k half-edges. We define the $k \times k$ matrix W by

$$W_{hh'} = \begin{cases} |\mathcal{X}_{r(h')}|/|\mathcal{X}_{h'}| - 1 & \overline{h} = h', \\ |\mathcal{X}_{r(h')}|/|\mathcal{X}_{h'}| & r(\overline{h}) = r(h'), \overline{h} \neq h', \\ 0 & r(\overline{h}) \neq r(h'). \end{cases}$$

Then

$$\zeta(u, \mathbb{X})^{-1} = \det(I_k - Wu).$$

Result: Three-Term Determinant Formula

Let $\mathbb{X}$ be a graph of groups with legs, A its twisted adjacency matrix, Q its twisted valency matrix. Then

$$\zeta(u, \mathbb{X})^{-1} = (1 - u^2)^{|E| - |V|}(1 + u)^{|L|} \det(I - Au - Qu^2),$$

where $L = L(X)$ is the set of legs of X.

Acknowledgements

We give special thanks to Professor Dmitry Zakharov for his invaluable mentorship and support, and to Professor Matthew Baker for organizing this project and funding it through his own research grant. This project was supported by NSF Research Grant DMS-2154224.

References

Bass, H. (1992). The Ihara–Selberg zeta function of a tree lattice. *International Journal of Mathematics*, 3(06):717–797.

Bass, H. (1993). Covering theory for graphs of groups. *J. Pure Appl. Algebra*, 89(1-2):3–47.

Meyer, M. and Zakharov, D. (2023). Chip-firing on graphs of groups. *arXiv preprint arXiv:2307.03348*.

Zakharov, D. (2021). Zeta functions of edge-free quotients of graphs. *Linear Algebra and its Applications*, 629:40–71.