Fourier Analysis and Dispersive Equations

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Background

- We consider the following one-dimensional cubic **nonlinear Schrödinger equation (NLS)**

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = |u|^2 u, \\ u(t_0, x) = u_*(x), \end{cases} \tag{1}$$

where $u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is the "unknown function," which we view as a complex-valued function of time $t \in \mathbb{R}$ and space $x \in \mathbb{R}$. We can equivalently view u as a map that sends a time t to a complex-valued function $u(t)$, where $u(t)$ sends a spatial point x to $u(t,x)$.

- Informally, the NLS is **dispersive** in the sense that solutions $u(t)$ become more "spread out" over time. In particular, the L^∞ norm of $u(t)$ tends to decay as $t \to \infty$.

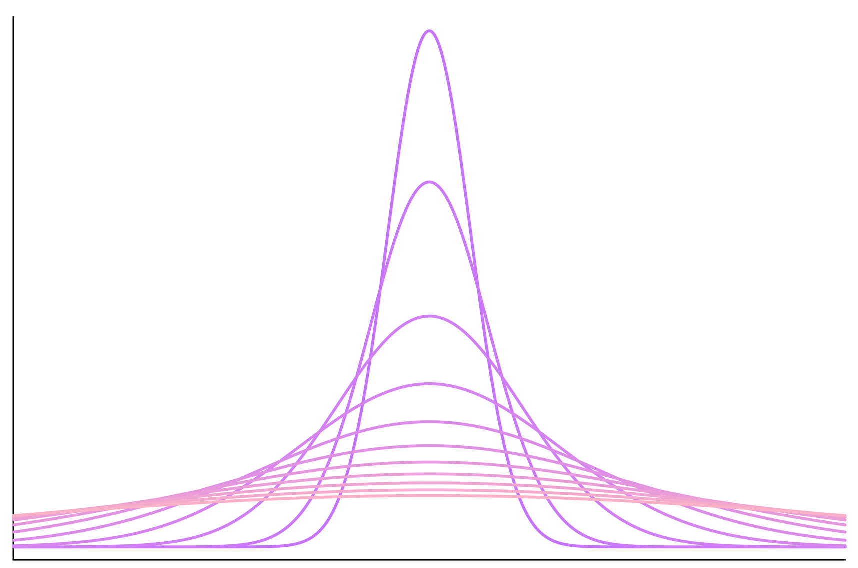

Figure 1. $|u(t,x)|^2$ vs. x for solution u to the Schrödinger equation; pink colors represent later times.

- Nonlinear Schrödinger equations play an important role in **quantum mechanics**, where they govern the behavior of **wave functions**. They also dictate the behavior of **Bose-Einstein condensates**; the 2001 Nobel Prize in Physics was awarded "for the achievement of Bose-Einstein Condensation" [4].

The Fourier Transform

The (one-dimensional) **Fourier transform** is an operator that sends a function $f : \mathbb{R} \to \mathbb{C}$ with sufficient decay in "physical space" to a function $\hat{f} : \mathbb{R} \to \mathbb{C}$ in "frequency space." If f is L^1,

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx.$$

- Many problems in physical space become significantly easier in frequency space, which makes the Fourier transform a useful tool.
- Operations in physical space correspond to different ones in frequency space, and vice versa:

physical space	frequency space
multiplication by polynomial	differentiation
multiplication by $e^{ix\xi_0}$	right translation by ξ_0
pointwise multiplication	convolution

- For functions f with sufficiently nice regularity and decay, we have the **inversion formula**

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} \, d\xi.$$

- By **Plancherel's theorem**, the L^2 norm of a function is preserved under the Fourier transform:

$$\int_{\mathbb{R}} |f(x)|^2 \, dx = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \, d\xi.$$

Example

Put $t_0 = 0$, and consider the linear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = 0.$$

Assuming that $u(t)$ is Schwartz, we can solve this equation using the Fourier transform. Taking Fourier transforms of both sides yields the ODE

$$\partial_t \widehat{u(t)}(\xi) = -\frac{i|\xi|^2}{2}\widehat{u(t)}(\xi) \quad \xrightarrow{\text{solving ODE}} \quad \widehat{u(t)}(\xi) = e^{-it|\xi|^2/2}\widehat{u_*}(\xi).$$

Applying the Fourier inversion formula yields

$$u(t,x) = \int_{\mathbb{R}} e^{i(-t|\xi|^2/2 + x\xi)} \widehat{u_*}(\xi) \, d\xi \tag{2}$$

as the solution to the linear equation.

Linear Propagators and Profiles

Motivated by Equation (2), we define the **linear propagator** $e^{it\partial_x^2/2}$ on Schwartz functions u_* by

$$e^{it\partial_x^2/2} u_*(x) := \int_{\mathbb{R}} e^{i(-t|\xi|^2/2 + x\xi)} \widehat{u_*}(\xi) \, d\xi.$$

Also, we define the **profile** of a solution $u(t,x)$ to be

$$f(t) := e^{-it\partial_x^2/2} u(t).$$

The profile recovers the initial data that would have produced the solution under linear evolution.

General Techniques

1. **Bootstrapping:** This is the continuous analogue of induction. We associate with each time $t \in \mathbb{R}$ a **hypothesis** $H(t)$ and a **conclusion** $C(t)$. If one can show that
   - $H(t)$ holds for at least one time t,
   - $H(t)$ implies $C(t)$,
   - $C(t)$ implies $H(t')$ for all t' in a neighborhood of t,
   - and the set of t where $C(t)$ holds is closed,

   then we can conclude that $C(t)$ holds for all times t.
2. **Local Existence Results:** If the initial datum u_* has sufficient regularity and decay, then there exists a finite time $T > 1$ and a unique solution u to the NLS on the interval $[0, T]$.
   - Furthermore, u has desirable regularity and decay properties.
   - This result is especially useful when applied in tandem with bootstrapping.
3. **Conservation Laws:** Many PDE have associated quantities (e.g., mass, energy, momentum) that are conserved over time, and these **conservation laws** are often useful in the analysis of these equations. Solutions $u(t)$ of the NLS enjoy conservation of mass (L^2 norm):

$$\frac{d}{dt} \int_{\mathbb{R}} |u(t,x)|^2 \, dx = 0.$$

   This aligns with the physical interpretation of $|u(t,x)|^2$ as a probability density.
4. **Duhamel's Formula:** This allows us to convert a differential equation into an **integral equation** that is often easier to handle. The integral equation for the above NLS is

$$u(t) = e^{i(t-t_0)\partial_x^2/2} u_* - i \int_{t_0}^{t} e^{i(t-s)\partial_x^2/2} |u(s)|^2 u(s) \, ds.$$

   This gives an easier approach to bound the size of quantities related to the solution $u(t)$.

Methods of Kato and Pusateri

The following result was shown by Hayashi and Naumkin in [1]:

Theorem (Informal Statement): Put $t_0 = 1$. Suppose u_* has a sufficiently nice regularity and decay. Then there exists a unique global solution u to the above NLS such that $||u(t)||_{L^\infty} \lesssim (1+|t|)^{-1/2}$. Furthermore, there exist bounded functions $W, \Phi : \mathbb{R} \to \mathbb{C}$ such that, as $t \to \infty$,

$$u(t,x) = (it)^{-1/2} W(x/t) \exp\left(i\frac{|x|^2}{2t} + i|W(x/t)|^2 \log(t) + i\Phi(x/t) \right) + O(t^{-1/2-\beta}) \tag{3}$$

uniformly in $x \in \mathbb{R}$, for some small $\beta > 0$.

- The notation $X \lesssim Y$ means that $X \le CY$ for some constant $C > 0$.
- Note how the L^∞ decay given by the theorem, along with the aforementioned conservation of L^2 norm, demonstrates the dispersive nature of the NLS.
- Hayashi and Naumkin work with a vector field approach involving the operator $J = x - it\nabla$.
- Kato and Pusateri were able to show the same results of Hayashi and Naumkin using a simpler analysis in frequency space [3]. The key idea is that if $u(t) = e^{it\partial_x^2/2} f(t)$, then

$$\|u(t)\|_{L^\infty} \lesssim t^{-1/2} \|\widehat{f(t)}\|_{L^\infty} + t^{-3/4+\epsilon} \|\partial_\xi \widehat{f(t)}\|_{L^2} \quad \text{for arbitrarily small } \epsilon > 0,$$

  so we can estimate these norms for the profile in frequency space instead.

Goal

Our goal (in progress) is to find functions $u(t,x)$ satisfying Equation (1) such that

$$\lim_{t\to\infty} \left(u(t) - (it)^{-1/2} e^{i|x|^2/2t} \widehat{u_+}(\cdot/t) \exp\left(-i|\widehat{u_+}(\cdot/t)|^2 \log(t) \right) \right) = 0$$

in L^2 under the conditions that the given final datum u_+ has sufficient decay.

This has already been shown by Hayashi and Naumkin in [2], but we are working on applying the methods of Kato and Pusateri to simplify the proof in a similar manner.

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References

[1] Nakao Hayashi and Pavel I. Naumkin. Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations. *American Journal of Mathematics*, 120(2):369–389, 1998.

[2] Nakao Hayashi and Pavel I. Naumkin. Domain and range of the modified wave operator for Schrödinger equations with a critical nonlinearity. *Communications in Mathematical Physics*, 267(2):477–492, October 2006.

[3] Jun Kato and Fabio Pusateri. A new proof of long-range scattering for critical nonlinear Schrödinger equations. *Differential and Integral Equations*, 24(9-10):923–940, September 2011.

[4] NobelPrize.org. The nobel prize in physics 2001. https://www.nobelprize.org/prizes/physics/2001/summary, 2001. Accessed: 14 July 2024.

[5] Terence Tao. *Nonlinear Dispersive Equations: Local and Global Analysis*. Number 106 in Regional Conference Series in Mathematics. American Mathematical Society, 2006.