To prove the stability of finite difference methods for approximating partial differential equations, it is necessary to be able to construct families of matrices whose powers are uniformly bounded. The Kreiss Matrix Theorem provides a condition that is equivalent to a family of matrices being uniformly power bounded. We state the version proven in [3].

Kreiss Matrix Theorem: Given a family $A$ of finite-dimensional matrices, the following are equivalent:

* There exists $C > 0$ such that $\| A^k \| \leq C k$ for all $k$. Then $A$ is power bounded.
* There exists $C > 0$ such that $\| \frac{1}{k} (A - I)^k \| \leq C k$ for all $k$ and $|k| > 1$.

A result of El-Fallah and Ransford [2] characterizes the relationship between $M(A)$ and $P(A)$ for operators that satisfy a resolvent condition like the Kreiss condition.

**Theorem:** Let $X$ be a complex Banach space and $A$ be a bounded linear operator on $X$. Suppose that $A$ satisfies a resolvent condition of the form $\| (A - \lambda I)^{-1} \| \leq C |\lambda|^{-\alpha}$

for some compact set $\Sigma \subseteq \mathbb{C}$ and some positive scalar $C$. If $k$ is finite, then

$\| A^k \| \leq C k^\alpha$.

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$\| A^k \| \leq C k^\alpha$.  

**Research Question**

My objective is to relate $M(A)$ and $P(A)$ for operators $A$ on an infinite dimensional Hilbert space. An important class of operators on an infinite dimensional space are Toeplitz operators. Thus, Toeplitz operators serve as a valuable source of examples. This project provides an in-depth view into operators of the form $A = T_\phi^* T_\psi$. Such operators are of interest because they are among the simplest nontrivial examples of operators. Namely, $f$ and $g$ can be easily selected so that they are not contractive, not normal, and not Toeplitz, so many of the standard results of operator theory are inapplicable. For certain operators of this form, we have the following result.

**Proposition:** Let $A = T_\phi^* T_\psi$. Then $P(A) \leq 1$, and analytic is $B$ such that $\| A \| = B$. Then, $P(A) \leq M(A)$.

I have extensively studied the cases where $f(z) = z$ and $f(z) = z^2$ and $g(z) = e + i k z$ and provide explicit upper and lower bounds for $M(A)$ and $P(A)$. I aim to provide a stronger result than that of El-Fallah and Ransford for these particular operators.

**Motivation**

**Methods**

- $M(A)$, $M(B)$: $\| |A| - |B| \| = 1$
- $\| \phi \|$ for $\| |A| - |B| \| = 1$
- RKHS properties allow point evaluation of inner products on $\mathbb{R}^2$ and speed up computations.
- $\| T_\phi \| = \| T_\psi \|$
- $\| \frac{1}{k} (A - I)^k \| \leq C k$ for all $k$ and $|k| > 1$.
- For purely analytic or conjugate analytic $f$,
  - $\| T_\phi \| = \| T_\psi \|$
  - $\| T_\phi \| = \| T_\psi \|$
- For a continuous function $f$ and $g$ analytic, $\| T_\phi \| = \| T_\psi \|$

**Results for $A = T_\phi^* T_\psi$**

1. $\| |A| - |B| \| \leq M(A) \| B \|$
2. $\| |A| - |B| \| \leq M(A) \| B \|$

where $\epsilon_0$ and $\epsilon_1$ are constant numbers.

**Ideas for Future Research**

A question of interest is whether there exists a sequence of operators $(A_n)_{n=1}^\infty$ such that $M(A_n) \leq n$ but $P(A_n) \to \infty$. Determining this will require some more experimentation with examples. I can consider more examples where $f(z)$ cannot be analytically continued to D. Some examples could be $f(z)$ that is the sum of analytic and conjugate analytic functions, or where $f(z)$ has a branch cut. A natural next step is the generalization of $g(z)$ to any polynomial lacking zeros on $\mathbb{D}$.

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**References**