Estimates for Power-Bounded Operators
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## Hardy Space Toeplitz Operators

Let $\mathbf{D} \subseteq \mathbf{C}$ denote the open unit disk and let $d \theta$ denote the normalized arc length measure on the unit circle $\partial \mathbf{D}$, i.e. $\theta(\partial \mathbf{D})=1$. The Hardy space $H^{2}$ is the closed linear span in $L^{2}(\partial \mathbf{D}, \theta)$ of $\left\{z^{n}: n \geq 0\right\}$ where $z$ is the coordinate function on $\partial \mathbf{D}$. For $f \in L^{\infty}(\partial \mathbf{D}, \theta)$, the Toeplitz operato with symbol $f$, denoted $T_{f}$, is the operator on $H^{2}$ defined by

$$
T_{f} h=P(f h)
$$

where $P$ denotes the orthogonal projection of $L^{2}(\partial \mathbf{D}, \theta)$ onto $H^{2}$.
On $\partial \mathbf{D}$, we let $\chi_{n}\left(e^{i t}\right)=e^{i n t}$. Observe that $\left\{\chi_{n}: n \in \mathbf{N}\right\}$ forms an orthonormal basis for $H^{2}$ and the matrix of a Toeplitz operator with respect to it. If $f$ is a function in $L^{\infty}(\mathbf{T})$ with Fourier coefficients $\hat{f}(n)=\int_{\partial \mathbf{D}} f \chi_{-n} d \theta$, then the matrix $\left\{a_{m, n}\right\}_{m, n \in \mathbf{N}}$ for $T_{f}$ with respect to the basis $\left\{\chi_{n}: n \in \mathbf{N}\right\}$ is

$$
a_{m, n}=\left(T_{f} \chi_{n}, \chi_{m}\right)=\int_{\partial \mathbf{D}} f \chi_{n-m} d \theta=\hat{f}(m-n)
$$

The matrix coefficients satisfy $a_{m, n}=a_{m-n, 0}$. Such a matrix is called a Toeplitz matrix.

## Power Bound and Resolvent Bound

Let $A$ be a bounded operator with spectrum $\sigma(A)$ contained within the closed unit disk $\overline{\mathbf{D}}$ of $\mathbf{C}$

$$
M(A):=\sup _{n \geq 0}\left\|A^{n}\right\|
$$

If $M(A)<+\infty$, then $A$ is power bounded. If $A$ is a contraction, then $M(A)=1$. If $\sigma(A) \subseteq \mathbf{D}$ then $A$ is power bounded. I investigate the relation between $M(A)$ and $P(A)$, where

$$
P(A):=\sup _{|\lambda|>1}^{|\lambda|} \operatorname{dist}(\lambda, \sigma(A))\left\|(\lambda I-A)^{-1}\right\|
$$

Diagonal operators illustrate the motivation for the expression $P(A)$. Let $D=\operatorname{diag}\left(d_{1}, \ldots, d_{n}\right)$. Then,

$$
\left\|(z I-D)^{-1}\right\|=\max _{j}\left|\left(z-d_{j}\right)^{-1}\right|=\frac{1}{\min _{j}\left|z-d_{j}\right|}=\frac{1}{\operatorname{dist}(z, \sigma(D))}
$$

Therefore, $P(D)=1$. Diagonal operators commute with their adjoint, i.e. are normal, so we have that $M(D)=1$.

## Reproducing Kernel Hilbert Spaces

## The exposition in this section is based on [4].

Definition: Given a set $X$, we call $\mathcal{H}$ a reproducing kernel Hilbert space (RKHS) over $\mathbf{C}$ provided
$\mathcal{H}$ is a vector subspace of $\mathcal{F}(X, \mathrm{C})$

- $\mathcal{H}$ is endowed with an inner product, making it into a Hilbert space.
- For every $y \in X$, the linear evaluation functional $E_{y}: \mathcal{H} \rightarrow \mathrm{C}$ defined by $E_{y}(f)=f(y)$ is bounded. The vector that corresponds to this evaluation map is called the kernel vector. An

The Hardy space $H^{2}$ is unitarily equivalent to the RKHS $H^{2}(\mathbf{D})$ over the open disk $\mathbf{D}$ in such a way that coordinate multiplication is preserved. The kernel vector for $H^{2}(\mathbf{D})$ for a point $\omega \in \mathbf{D}$ is

$$
g(z)=\sum_{n=0}^{\infty} \overline{\omega^{n}} z^{n}=\frac{1}{1-\bar{w} z}
$$

because for any $f(z)=\sum_{n=0}^{\infty} a_{n} z^{n} \in H^{2}(\mathbf{D})$, the statement $\langle f, g\rangle=f(\omega)$ holds. For Toeplitz
operators, we have the following property. operators, we have the following property.
Proposition: If $q$ is analytic and bounded, then $T_{q}^{*} k_{\omega}=\overline{q(\omega)} k_{\omega}$ where $k_{\omega}$ is the kernel vector corresponding to $\omega \in \mathrm{D}$

## Motivation

To prove the stability of finite difference methods for approximating partial differential equations, it is necessary to be able to construct families of matrices whose powers are uniformly bounded The Kreiss Matrix Theorem provides a condition that is equivalent to a family of matrices being uniformly power bounded. We state the version proven in [3].
Kreiss Matrix Theorem: Given a family $\mathcal{F}$ of (finite-dimensional) matrices, the following are equiv alent

- There exists $C>0$ such that $M(A) \leq C$ for all $A \in \mathcal{F}$
- There exists $C>0$ such that $\left\|(\lambda I-A)^{-1}\right\| \leq C(|\lambda|-1)^{-1}$ for all $A \in \mathcal{F}$ and $|\lambda|>1$.

A result of El-Fallah and Ransford [2] characterizes the relationship between $M(A)$ and $P(A)$ for operators that satisfy a resolvent condition like the Kreiss condition.
Theorem: Let $X$ be a complex Banach space and $A$ be a bounded linear operator on $X$. Suppose that $A$ satisfies a resolvent condition of the form

$$
\left\|(\lambda I-A)^{-1}\right\| \leq \frac{C}{\operatorname{dist}(\lambda, E)}
$$

for some compact set $E \subseteq \partial \mathbf{D}$ and some positive scalar $C$. If $E$ is finite, then

$$
\sup _{n \geq 0}\left\|A^{n}\right\| \leq \frac{e}{2} C^{2} \# E
$$

## Research Question

My objective is to relate $M(A)$ and $P(A)$ for operators $A$ on an infinite dimensional Hilbert space. An operators. Thus, Toeplitz operators serve as a valuable source of examples.
This project provides an in-depth view into operators of the form $A=T_{g}^{-1} T_{f} T_{g}$. Such operators are of interest because they are among the simplest nontrivial examples of operators. Namely so many of the standard results of operator theory are inapplicable. For certain operators of this form, we have the following result.
Proposition: Let $A=T_{g}^{-1} T_{f} T_{g},\|f\|_{\infty} \leq 1$, and $g$ analytic in $\overline{\mathbf{D}}$ such that $\sigma(A)=\overline{\mathbf{D}}$. Then, $P(A) \leq M(A)$
I have extensively studied the cases where $f(z)=\bar{z}$ and $f(z)=\frac{z+\bar{z}}{2}$ and $g(z)=a+b z$ and provide explicit upper and lower bounds for $M(A)$ and $P(A)$. I aim to provide a stronger result than that of El-Fallah and Ransford for these particular operators.

## Methods

- $\left|\left\langle B h, h^{\prime}\right\rangle\right| \leq\|B\|$ for $\|h\|=\left\|h^{\prime}\right\|=1$
- RKHS properties allow point evaluation of inner products on $H^{2}(\mathbf{D})$ and speed up computations.
computa
$-T_{\theta}^{*}=T_{\bar{\varphi}}$
- If $\varphi \in L^{\infty}(\partial \mathbf{D})$ and $\psi$ and $\bar{\theta}$ are functions in $H^{\infty}$, then $T_{\varphi} T_{\psi}=T_{\varphi \psi}$ and $T_{\theta} T_{\varphi}=T_{\theta \varphi}$. - For purely analytic or conjugate analytic $f$,

$$
T_{\frac{1}{\lambda-f}}=\left(\lambda I-T_{f}\right)^{-1}
$$

$T_{g^{-1}} T_{f} T_{g}=T_{f}+T_{g^{-1}}\left[T_{f}, T_{g}\right]$ where $\left[T_{f}, T_{g}\right]$ is the commutator. Since $f$ is bounded and $g$ is analytic on $\overline{\mathbf{D}}$, it follows that $\left[T_{f}, T_{g}\right]$ is compact.
If $\varphi$ is a continuous function on $\partial \mathbf{D}$, then $\sigma\left(T_{\varphi}\right)=\mathcal{R}(\varphi) \cup\{\lambda \in \mathbf{C}: i(\varphi, \lambda) \neq 0\}$ where $i(\varphi, \lambda)$ is the winding number of $\varphi$ around $\lambda$.

Results for $A=T_{g}^{-1} T_{\bar{z}} T_{g}$

$$
\begin{aligned}
& \max \left\{1, \frac{|b|}{\sqrt{|a|^{2}-|b|^{2}}}\right\} \leq M(A) \leq \min \left\{\frac{|a|+|b|}{|a|-|b|^{2}}, 1+\frac{|b|}{\sqrt{|a|^{2}-|b|^{2}}}\right\} \\
& \max \left\{1, \frac{c_{0}|b|}{\sqrt{|a|^{2}-|b|^{2}}}\right\} \leq P(A) \leq \min \left\{M(A), 1+\frac{c_{c}|b|}{\sqrt{|a|^{2}-|b|^{2}}}\right\}
\end{aligned}
$$

where $c_{0}$ and $c_{1}$ are numerical constants.
Results for $A=T_{g}^{-1} T_{\frac{t+\pi}{2}} T_{g}$
Let $P_{S}(A):=\sup _{|\lambda|>1} \operatorname{dist}(\lambda, S)| |(\lambda I-A)^{-1} \|$. Then,

- $P_{\sigma(A)}(A) \leq P_{\{-1,1\}}(A)$
- $M(A) \leq e P_{\{-1,1\}}(A)^{2} \leq 2 e P_{\sigma(A)}(A)^{2}$

The first result above follows from the definition of $P_{S}(A)$ and the second follows from the result of El-Fallah and Ransford and plane geometry. Below, I provide explicit bounds for $M(A)$ and $P(A)$.

$$
\begin{gathered}
\frac{1}{2}\left|\frac{b}{a}\left(\frac{|a|}{\sqrt{|a|^{2}}-|b|^{2}}\right)-\frac{\bar{b}}{\bar{a}}\left(\frac{\sqrt{|a|^{2}-|b|^{2}}}{|a|}\right)\right| \leq M(A) \leq \frac{|a|+|b|}{|a|-|b|} \\
\max \left\{1, \sqrt{\frac{M(A)}{2 e}}\right\} \leq P(A) \leq \frac{|a|+|b|}{|a|-|b|}
\end{gathered}
$$

## Ideas for Future Research

A question of interest is whether there exists a sequence of operators $\left\{A_{i}\right\}_{i=1}^{\infty}$ such that $M\left(A_{i}\right)<$ $\infty$ but $P\left(A_{i}\right) \rightarrow \infty$. Determining this will require some more experimentation with examples. We can consider more examples where $f(z)$ cannot be analytically continued to $\mathbf{D}$. Some examples could be $f(z)$ that is the sum of analytic and conjugate analytic functions or where $f(z)$ has a
branch cut. A natural next step is the generalization of $g(z)$ to any polynomial lacking zeros on $\overline{\mathbf{D}}$.

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