Erdös-Szekeres Polynomials and Their $L_{2}$ Norms
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What are Erdős-Szekeres Polynomials?
Erdős-Szekeres Polynomials are polynomials of the form

$$
\prod_{i=1}^{n}\left(1-z^{s_{j}}\right)=a_{0}+a_{1} z+\ldots+a_{d} z^{d}
$$

where the $s_{j}$ 's are positive integers. We define the $L_{2}$ norm by

$$
\|P(z)\|_{2}=\left(\frac{1}{2 \pi} \int_{0}^{2 \pi}|P(z)|^{2} d z\right)^{1 / 2}=\sqrt{a_{0}^{2}+a_{1}^{2}+\ldots+a_{d}^{2}}
$$

Example: Let $P(z)=(1-z)\left(1-z^{2}\right)\left(1-z^{3}\right)\left(1-z^{4}\right)=z^{10}-z^{9}-z^{8}+2 z^{5}-z^{2}-z+$

1. $\operatorname{deg}(P)=1+2+3+4=10$
2. $\|P(z)\|_{2}=\sqrt{10}$
3. Each $a_{k}$ is the difference between the number of ways to choose an even and odd number of $s_{j}$ terms which sum to $k$

Plotting the norms
In order to estimate the distribution of L 2 norms for fixed n , we generate random polynomials using a Monte Carlo algorithm to sample from the set of all possible exponents and then plot the using a Monte Carlo algorithm to s.


While this is only a single fixed value of n with exponents bounded by M , the bell curve above is observed for other values of $n \leq M$ as well.

## W/hat can we say about the distribution?

Let $A_{2}(M, n)$ be the average of the $L_{2}$ norm square of Erdős-Szekeres polynomials from all $n$ tuples ( $s_{1}, \cdots, s_{n}$ ) such that $1 \leq s_{j} \leq M$ for $1 \leq j \leq n$. Let $V_{2}(M, n)$ be the associated variance.

Analytic Results

## Theorem 1

Theorem 1
For fixed $n$ we have

$$
\lim _{M \rightarrow \infty} A_{2}(M, n)=2^{n} \text { and } \lim _{M \rightarrow \infty} V_{2}(M, n)=0
$$

## Theorem 2

Write $V_{2}(M, n)=\left(B_{2}(M, n)-A_{2}(M, n)^{2}\right)^{1 / 2}$. For $k \geq 1$, let $n_{k} \geq 1$ and $M_{k} \geq 1$. Assume that $n_{k} \rightarrow \infty$ and $M_{k} \rightarrow \infty$ as $k \rightarrow \infty$, in such a way tha

$$
\lim _{k \rightarrow \infty} M_{k}^{1 / n_{k}}=\rho \in[1, \infty] .
$$

Let $s_{0} \in\left(\pi, \frac{3}{2} \pi\right)$ be the unique root in $\left(\pi, \frac{3}{2} \pi\right)$ of the equation $\tan s=s$. Then,

$$
\begin{gathered}
\lim _{k \rightarrow \infty} A_{2}\left(M_{k}, n_{k}\right)^{1 / n_{k}}=2 \max \left\{1, \frac{1}{\rho}\left(1-\frac{\sin s_{0}}{s_{0}}\right)\right\} \\
\lim _{k \rightarrow \infty} B_{2}\left(M_{k}, n_{k}\right)^{1 / n_{k}}=\max \left\{\frac{8}{\rho^{2}}, \frac{6}{\rho}, 4\right\} .
\end{gathered}
$$

3. If $\rho<\frac{3}{2}$, then

$$
\lim _{k \rightarrow \infty} V_{2}\left(M_{k}, n_{k}\right)^{1 / n_{k}}=\sqrt{\max \left\{\frac{8}{\rho^{2}}, \frac{6}{\rho}\right\} .}
$$

A Sketch of the Proof
We first change the order of summation and integration. Then we study the asymptotic behavior We first change the order of summation and integ
of the integral through the following methods:

1. Hölder's inequality
2. Fatou's lemma
3. Results from uniform distribution
4. Bounding and estimating the integrand on different domains

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Minimizing the $L_{2}$ Norm

## - Given an $n$, what is the minimum $L_{2}$ norm?

-What polynomial has this norm?
We used an algorithm adapted from Maltby [2] and a greedy algorithm to try and answer these questions.


Where $D=\sum_{i=1}^{n-1} s_{i}$. Here is a plot of the growth of the $L_{2}$ norm of the polynomials generated




