Georgia Tech

What are Erdős-Szekeres Polynomials?

Erdős-Szekeres Polynomials are polynomials of the form

$$\prod_{j=1}^{n} (1 - z^{s_j}) = a_0 + a_1 z + \dots + a_d z^d$$

where the s_i 's are positive integers. We define the L_2 norm by

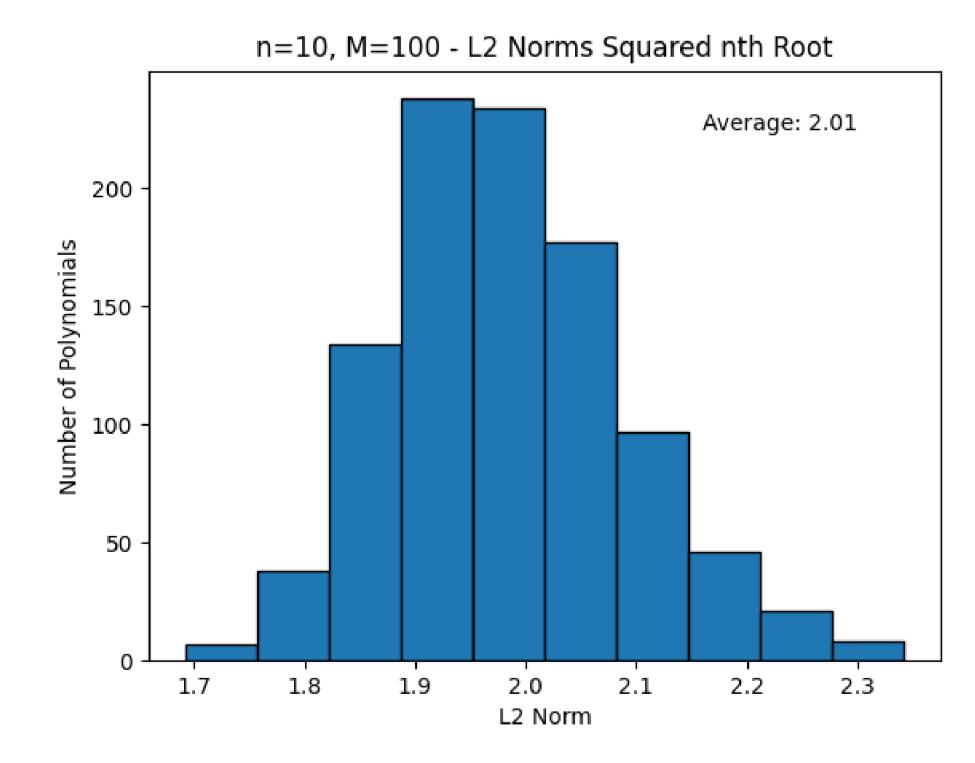
$$||P(z)||_{2} = \left(\frac{1}{2\pi} \int_{0}^{2\pi} |P(z)|^{2} dz\right)^{1/2} = \sqrt{a_{0}^{2} + a_{1}^{2} + \dots + a_{d}^{2}}$$

Example: Let $P(z) = (1-z)(1-z^2)(1-z^3)(1-z^4) = z^{10} - z^9 - z^8 + 2z^5 - z^8$

- 1. deg(P) = 1 + 2 + 3 + 4 = 10
- 2. $||P(z)||_2 = \sqrt{10}$
- 3. Each a_k is the difference between the number of ways to choose an even and odd number of s_{i} terms which sum to k

Plotting the norms

In order to estimate the distribution of L2 norms for fixed n, we generate random polynomials using a Monte Carlo algorithm to sample from the set of all possible exponents and then plot the nth root of the L2 norms squared.



While this is only a single fixed value of n with exponents bounded by M, the bell curve above is observed for other values of $n \leq M$ as well.

Erdős-Szekeres Polynomials and Their L_2 Norms

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What can we say about the distribution?

Let $A_2(M, n)$ be the average of the L_2 norm square of Erdős-Szekeres polynomials from all ntuples (s_1, \dots, s_n) such that $1 \le s_j \le M$ for $1 \le j \le n$. Let $V_2(M, n)$ be the associated variance.

Analytic Results

Theorem 2

For fixed *n* we have

$$\lim_{M \to \infty} A_2(M, n) = 2^n \text{ and } \lim_{M \to \infty}$$

$$z^2 - z + 1$$

$$\lim_{k \to \infty} B_2(M_k, n_k)^{1/n_k} = \max$$

3. If $\rho < \frac{3}{2}$, then

$$\lim_{k \to \infty} V_2(M_k, n_k)^{1/n_k} = \sqrt{\max\left\{\frac{8}{\rho^2}, \frac{6}{\rho}\right\}}.$$

A Sketch of the Proof

We first change the order of summation and integration. Then we study the asymptotic behavior of the integral through the following methods:

- 1. Hölder's inequality
- 2. Fatou's lemma
- 3. Results from uniform distribution
- 4. Bounding and estimating the integrand on different domains

 $n_k \to \infty$ and $M_k \to \infty$ as $k \to \infty$, in such a way that

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 $V_2(M,n) = 0.$

Write $V_2(M, n) = (B_2(M, n) - A_2(M, n)^2)^{1/2}$. For $k \ge 1$, let $n_k \ge 1$ and $M_k \ge 1$. Assume that

 $\lim_{k \to \infty} M_k^{1/n_k} = \rho \in [1, \infty].$

Let $s_0 \in (\pi, \frac{3}{2}\pi)$ be the unique root in $(\pi, \frac{3}{2}\pi)$ of the equation $\tan s = s$. Then,

$$\lim_{k \to \infty} A_2(M_k, n_k)^{1/n_k} = 2 \max\left\{1, \frac{1}{\rho}\left(1 - \frac{\sin s_0}{s_0}\right)\right\}$$
$$\lim_{k \to \infty} B_2(M_k, n_k)^{1/n_k} = \max\left\{\frac{8}{\rho^2}, \frac{6}{\rho}, 4\right\}.$$

- Given an n, what is the minimum L_2 norm?
- What polynomial has this norm?

We used an algorithm adapted from Maltby [2] and a greedy algorithm to try and answer these questions.

Suppose that $\{s_j\} = \{a, b\}$.

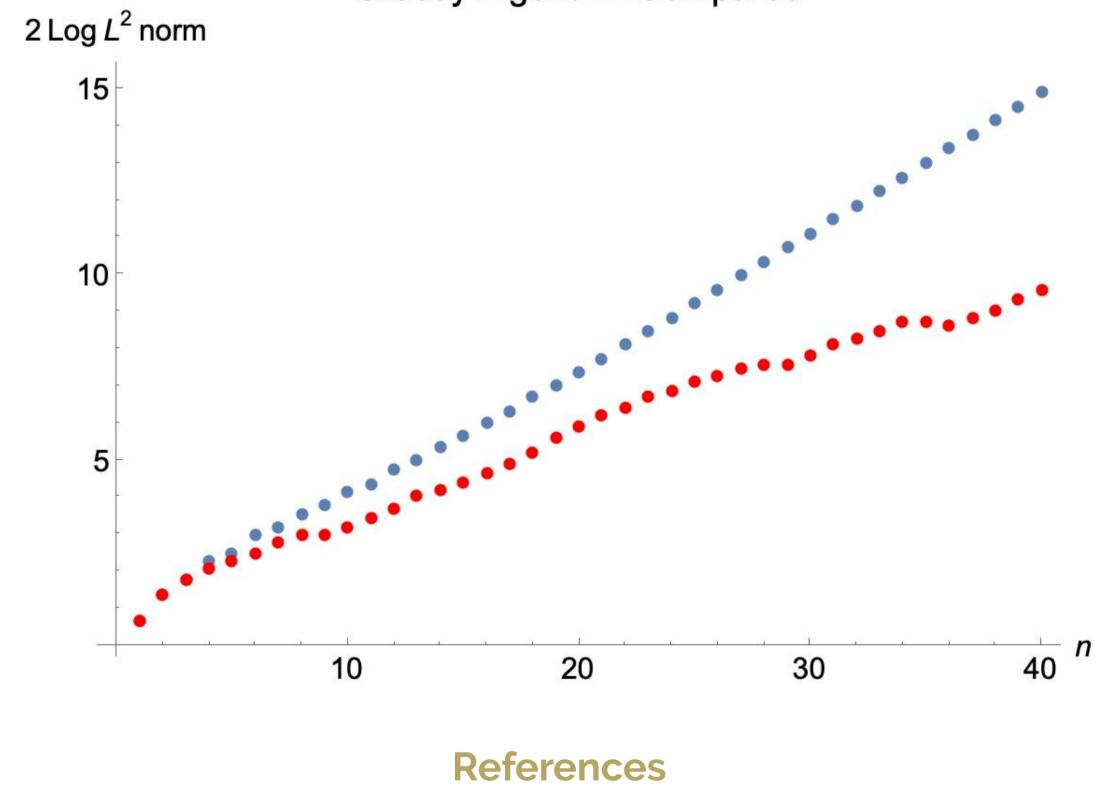
 $0 < a < b < a + b \implies P(\{a, b\})$ $0 < a = b < a + b \implies P(\{a, b\})$

Using this recurre

$$C(S_n, k) = C(S_{n-1}, k) - C(S_{n-1}, k - s_n)$$

rence we express the L_2 norm of $P(S_n; z)$ as,
$$||P(S_n; z)||_2^2 = 2||P(S_{n-1}; z)||_2^2 - 2\sum_{k=0}^D C(S_{n-1}, k)C(S_{n-1}, k - s_n)$$

Greedy Algorithm Compared



[1] Paul Erdős and George Szekeres. On the product $\prod_{k=1}^{n}(1-z^{a_k})$. Publications de Lílnstitut Mathematique, pages 29–34, 1959. [2] Roy Maltby. Pure product polynomials and the prouhet-tarry-escott problem. Mathematics of Computation, 66(219):1323–1340, 1997.



Minimizing the L_2 Norm

Maltby Algorithm

$;z) = 1 - z^a - z^b + z^{a+b}$	$\implies \ P\left(\{a,b\};z\right)\ _2 = 2$
$;z) = 1 - 2z^a + z^{a+b}$	$\implies \ P(\{a,b\};z)\ _2 = \sqrt{6}$

A Greedy Algorithm

Given an n-tuple, $S_n = (s_1, s_2...s_n)$ of exponents, the coefficient of z^k is given by

Where $D = \sum_{i=1}^{n-1} s_i$. Here is a plot of the growth of the L_2 norm of the polynomials generated