What are Erdős-Szekeres Polynomials?

Erdős-Szekeres Polynomials are polynomials of the form
\[ \prod_{j=1}^{n} (1 - x^j) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} \]
where the \(a_j\)’s are positive integers. We define the \(L_2\) norm by
\[ ||P(z)||_2 = \left( \frac{1}{M} \int_{-1}^{1} |P(z)|^2 dz \right)^{1/2} \]

Example: Let \( P(z) = 1 - z - z^3(1 - z^2) = 1 - z^2 - z^4 + z^6 + z^8 - z + 1 \)
1. \( \deg(P) = 1 + 2 + 3 + 4 + 6 = 10 \)
2. \( ||P(z)||_2 = \sqrt{10} \)
3. \( |x_3 - x_2| = 2 \)

Plotting the norms

In order to estimate the distribution of \(L_2\) norms for fixed \(n\), we generate random polynomials using a Monte Carlo algorithm to sample from the set of all possible exponents and then plot the \(n\)th root of the \(L_2\) norms squared.

While this is only a single fixed value of \(n\) with exponents bounded by \(M\), the bell curve above is observed for other values of \(n \leq M\) as well.

What can we say about the distribution?

Let \(A_j(M, n)\) be the average of the \(L_2\) norm square of Erdős-Szekeres polynomials from all \(n\)-tuples \((s_1, \ldots, s_n)\) such that \(1 \leq s_j \leq M\) for \(1 \leq j \leq n\). Let \(V_j(M, n)\) be the associated variance.

Analytic Results

Theorem 1
For fixed \(n\) we have:
\[ \lim_{M \to \infty} A_j(M, n) = 2^n \text{ and } \lim_{M \to \infty} V_j(M, n) = 0. \]

Theorem 2
Write \(V_j(M, n) = (B_j(M, n) - A_j(M, n))^{1/2}\). For \(k \geq 1\), let \(M_k \geq 1\) and \(M_k \geq 1\). Assume that \(a_k \to \infty\) and \(M_k \to \infty\) as \(k \to \infty\), in such a way that
\[ \lim_{k \to \infty} \frac{M_k^{1/n_k}}{\rho} = \rho \in [1, \infty]. \]
1. Let \(x_j \in (\rho, 1/\rho)\) be the unique root in \((\rho, 1/\rho)\) of the equation \(x/n = \rho\). Then,
\[ \lim_{k \to \infty} \frac{B_j(M_k, n_k)^{1/n_k}}{M_k^{1/n_k}} = \max \left\{ 1, \frac{\rho - 1}{\rho \rho - 1} \right\}. \]
2. \[ \lim_{k \to \infty} B_j(M_k, n_k)^{1/n_k} = \max \left\{ 1, \frac{\rho - 1}{\rho \rho - 1} \right\}. \]
3. If \(\rho < \frac{1}{2}\), then
\[ \lim_{k \to \infty} B_j(M_k, n_k)^{1/n_k} = \frac{\rho}{\rho + 1}. \]

A Sketch of the Proof

We first change the order of summation and integration. Then we study the asymptotic behavior of the integral through the following methods:
1. Hölder’s inequality
2. Fatou’s lemma
3. Results from uniform distribution
4. Bounding and estimating the integrand on different domains

We used an algorithm adapted from Maltby [2] and a greedy algorithm to try and answer these questions.

Maltby Algorithm

Suppose that \(\{x_j\} = \{a, b\}\).
\[ 0 < a < b < a + b \quad \Rightarrow \quad P^*(\{a, b\}, z) = 1 - z^a - z^{b + k} \quad \Rightarrow \quad ||P(\{a, b\}, z)||_2 = 2 \]
\[ 0 < a = b < a + b \quad \Rightarrow \quad P^*(\{a, b\}, z) = 1 - 2z^a + z^{b + k} \quad \Rightarrow \quad ||P(\{a, b\}, z)||_2 = \sqrt{2} \]

A Greedy Algorithm

Given an \(n\)-tuple, \(S_n = (x_1, x_2, \ldots, x_n)\) of exponents, the coefficient of \(z^k\) is given by
\[ C(S_n, k) = C(S_{n-1}, 0) - C(S_{n-1}, k - n). \]
Using this recurrence we express the \(L_2\) norm of \(P(S_n, z)\) as,
\[ ||P(S_n, z)||^2 = 2 ||P(S_{n-1}, z)||^2 + 2 \sum_{k=0}^{n-1} C(S_{n-1}, k)C(S_{n-1}, k - n). \]

Where \(B = \sum_{k=0}^{n-1} x_k\). Here is a plot of the growth of the \(L_2\) norm of the polynomials generated.

Minimizing the \(L_2\) Norm

- Given an \(n\), what is the minimum \(L_2\) norm?
- What polynomial has this norm?

Acknowledgements

Thanks to our mentor, Professor Doron Lubinsky, for his support, as well as the Georgia Institute of Technology School of Mathematics and the National Science Foundation.

References