# Enumerating Acyclic Orientations of Complete Multipartite Graphs 

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## Objectives

We study acyclic orientations (AOs) of complete multipartite graphs. Our results include:

Explicit formulas for the number of AOs of complete multipartite graphs (asked in [1]). Relating AOs of complete bipartite graphs to permutations with a prescribed exceedance set.

## Introduction

An orientation of an undirected graph is an as signment of a direction to each of its edges. For graph $G$, let $\mathcal{A}(G)$ be the set of orientations which contain no directed cycle.


We denote by $\mathcal{A}(G, q)$ the set of AOs of $G$ with unique sink at a chosen vertex $q$;


Enumerating these sets is of interest in the mathematical community:

- $|\mathcal{A}(G)|=\left|\chi_{G}(-1)\right|=T_{G}(2,0)\left(T_{G}\right.$ is Tutte poly.)
- $|\mathcal{A}(G, q)|=\left|a_{1}\left(\chi_{G}\right)\right|=T_{G}(1,0)$
- \#P-complete, unknown approximability
as well as in other fields of study:
- $|\mathcal{A}(G, q)|$ counts branched polymers [3].
- $|\mathcal{A}(G, q)|$ gives the Ursell function of statistical physics [2].


## Complete Multipartite Graphs

The complete bipartite graph $K_{m, n}$ is the graph on two vertex sets whose edges are all those between the vertex sets:


Analogously, the complete $N$-partite graph $K_{n_{1}, \ldots, n_{N}}$ has $N$ vertex sets, and all edges between vertex sets:

## $K_{2,2,2}$



Question: What is $\left|\mathcal{A}\left(K_{m, n}\right)\right|$ ? $\left|\mathcal{A}\left(K_{n_{1}, \ldots, n_{N}}\right)\right|$ ?

## Partially Unlabeled

It is useful to instead consider $K_{m, n}^{\prime}$, the complete bipartite graph with vertices in the $n$-set unlabeled:


Now, $\left|\mathcal{A}\left(K_{m, n}^{\prime}\right)\right|$ can be counted (up to isomorphism within unlabeled vertex set) by counting 'canonical' topological sorts of the vertices:

$g g g \rightarrow g b_{1} g b_{2} g$

This idea is easily generalized to $K_{n_{1}, \ldots, n_{N}}^{\prime}$

## Partially Unlabeled Result

We have the following counts for AOs of partially unlabeled complete multipartite graphs:

$$
\left|\mathcal{A}\left(K_{n_{1}, \ldots, n_{N}}^{\prime}\right)\right|=\left(1+\sum_{i=2}^{N} n_{i}\right)^{n_{1}}\binom{\sum_{i=2}^{N} n_{i}}{n_{2}, \ldots, n_{N}} \quad \text { in particular } \quad\left|\mathcal{A}\left(K_{m, n}^{\prime}\right)\right|=(1+n)^{m}
$$

## Relabeling

Now, we might hope to count $\left|\mathcal{A}\left(K_{m, n}\right)\right|$ by counting the ways to relabel the unlabeled vertices in $K_{m, n}^{\prime}$ Purely counting the number of orientation/labeling pairs gives a multiset

$$
|\mathcal{L}|=n!(1+n)^{m}
$$

But this overcounts AOs of $K_{m, n}$. e.g.


Explicit formula for $K_{m, n}$

$$
\left|\mathcal{A}\left(K_{m, n}\right)\right|=\sum_{j=0}^{n-1}(-1)^{j} \cdot(1+n-j)^{m} \cdot(n-j)!\cdot S(n, n-j)
$$

and a similar formula can be given for $K_{n_{1}, \ldots, n_{N}}$ in terms of $N-1$ sums over parts $n_{2}, \ldots, n_{N}$.

## AOs and Permutations

The exceedance set of a permutation $\sigma \in S_{m+n}$ is $\operatorname{ex}(\sigma)=\{i \mid \sigma(i)>i\}$. Then, letting

$$
T(m, n)=\left\{\sigma \in S_{m+n} \mid \operatorname{ex}(\sigma)=\{1, \ldots, m\}\right\}
$$

we give a simple bijection between $\mathcal{A}\left(K_{m+1, n}, q\right)$ (with $q$ in the $(m+1)$-set) and $T(m, n)$ :

- Remove $q$, giving the set of AOs of $K_{m, n}$ with no sink in the $m$-set
© Effectively equate (canonical) topological sorts with (canonical) cycle decompositions
For example, if $m=3$ and $n=3$ :

gives the topological sort/cycle decomposition

$$
b_{3} g_{2} b_{1} b_{2} g_{3} g_{1} \longleftrightarrow 351264 \longleftrightarrow \text { (35)(1264) }
$$

## Conclusions/Future Questions

Our formulas gives ways to quickly compute the number of AOs for complete multipartite graphs. The idea of first considering a partially unlabeled graph might be useful in other problems.

- Which $G$ maximize $|\mathcal{A}(G)|$ for fixed $|V|,|E|$ ? [1] - Similar bijection for complete multipartite case? - Count \#AOs for grid graphs and hypercubes?


## References

[^0]
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