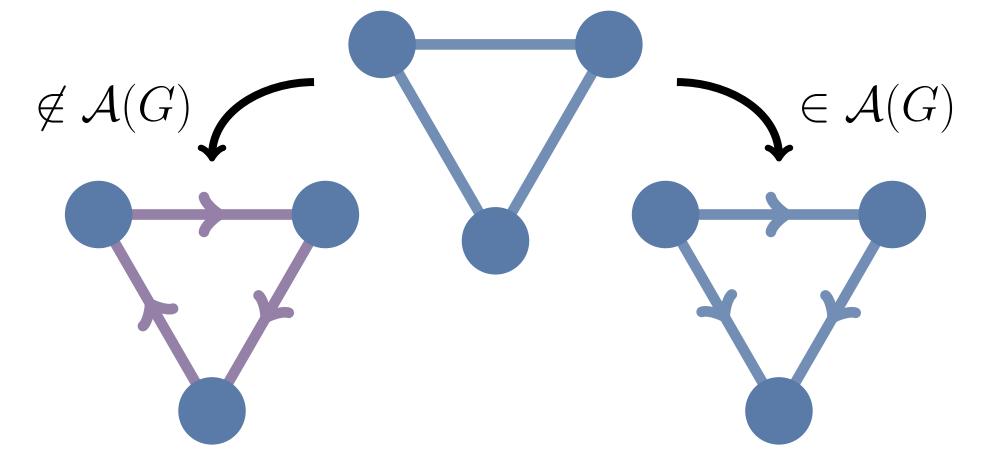
# Objectives

We study acyclic orientations (AOs) of complete multipartite graphs. Our results include:

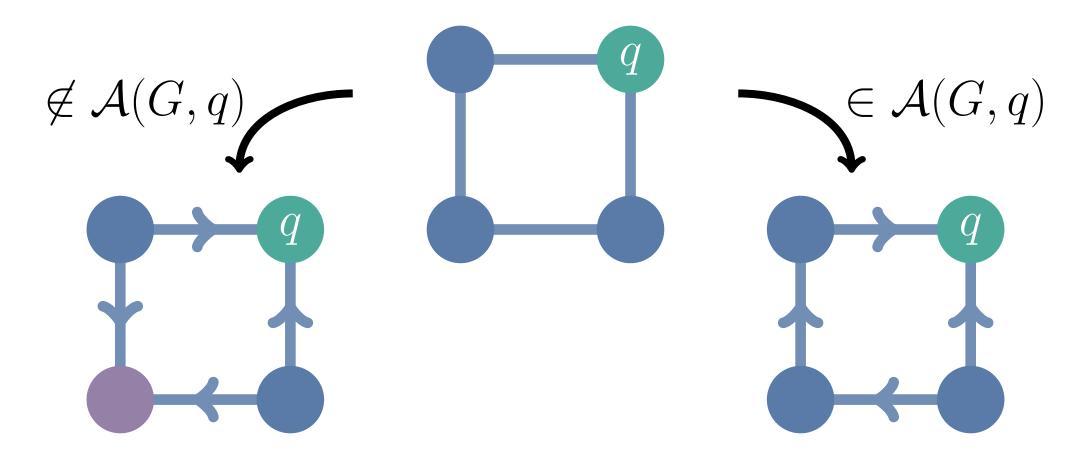
- Explicit formulas for the number of AOs of complete multipartite graphs (asked in [1]).
- Relating AOs of complete *bipartite* graphs to permutations with a prescribed exceedance set.

#### Introduction

An **orientation** of an undirected graph is an assignment of a direction to each of its edges. For graph G, let  $\mathcal{A}(G)$  be the set of orientations which contain no directed cycle.



We denote by  $\mathcal{A}(G,q)$  the set of AOs of G with unique sink at a chosen vertex q:



Enumerating these sets is of interest in the mathematical community:

- $|\mathcal{A}(G)| = |\chi_G(-1)| = T_G(2,0)$  ( $T_G$  is Tutte poly.)
- $|\mathcal{A}(G,q)| = |a_1(\chi_G)| = T_G(1,0)$
- $\#\mathcal{P}$ -complete, unknown approximability

as well as in other fields of study:

- $|\mathcal{A}(G,q)|$  counts branched polymers [3].
- $|\mathcal{A}(G,q)|$  gives the Ursell function of statistical physics [2].

# **Enumerating Acyclic Orientations of Complete Multipartite Graphs**

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#### **Complete Multipartite Graphs**

The <b>complete bipartite graph</b> $K_{m,n}$ is the graph on two vertex sets whose edges are all those between the vertex sets:	It bip
$m = 2$ $b_1$ $b_2$ $K_{2,3}$	
$n = 3$ $g_1$ $g_2$ $g_3$ Analogously, the complete N-partite graph $K_{n_1,,n_N}$ has N vertex sets, and all edges between vertex sets:	No wit
$K_{2,2,2}$ $K_{2$	toj
Question: What is $ \mathcal{A}(K_{m,n}) $ ? $ \mathcal{A}(K_{n_1,,n_N}) $ ?	Th

# Partially Unlabeled Result

We have the following cour		<b>–</b> 0	
$\left \mathcal{A}(K_{n_1,\ldots,n_N}')\right  = \left(1\right)$	$1 + \sum_{i=2}^{N} n_i \right)^{n_1}$	$\begin{pmatrix} \sum_{i=2}^N n_i \\ n_2, \dots, n_i \end{pmatrix}$	N) j

### Relabeling

	<b>.</b>
Now, we might hope to count $ \mathcal{A}(K_{m,n}) $ by counting	We
the ways to relabel the unlabeled vertices in $K'_{m,n}$ .	tat
Purely counting the number of orientation/labeling	WO
pairs gives a multiset	We
$ \mathcal{L}  = n!(1+n)^m$	exc
But this overcounts AOs of $K_{m,n}$ . e.g.	
$b_1$ $b_2$ $b_1$ $b_2$	it (
$+ x \chi + = + x \chi +$	$J\subseteq$

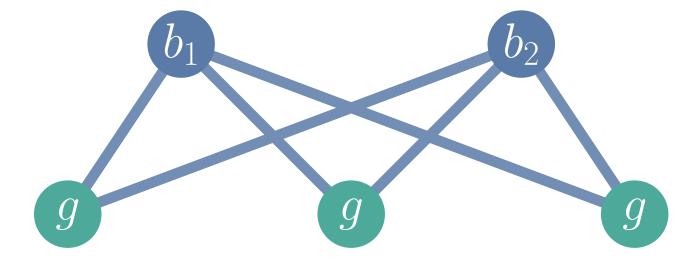
#### Explicit formula for $K_{m,n}$

$$|\mathcal{A}(K_{m,n})| = \sum_{j=0}^{n-1} (-1)^j \cdot (1+n-j)^m \cdot (n-j)! \cdot S(n,n-j)$$

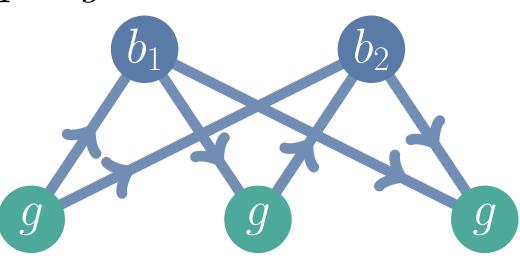
and a similar formula can be given for  $K_{n_1,\ldots,n_N}$  in terms of N-1 sums over parts  $n_2,\ldots,n_N$ .

# Partially Unlabeled

is useful to instead consider  $K'_{m,n}$ , the complete ipartite graph with vertices in the n-set unlabeled:



Iow,  $|\mathcal{A}(K'_{m,n})|$  can be counted (up to isomorphism) ithin unlabeled vertex set) by counting 'canonical' opological sorts of the vertices:



$$ggg \rightarrow gb_1gb_2g$$

his idea is easily generalized to  $K'_{n_1,\ldots,n_N}$ .

led complete multipartite graphs:

$$4(K'_{m,n})| = (1+n)^n$$

### **Inclusion-Exclusion**

Ve can count the number of 'non-canonical' orienation/labeling pairs in  $\mathcal{L}$ . Subtracting these out vould yield  $|\mathcal{A}(K_{m,n})|$  as desired.

Ve can achieve this with the principle of inclusionxclusion. Letting the 'bad sets' be

 $L_i = \{ (\mathcal{O}, \ell) \in \mathcal{L} \mid g_i \text{ non-canonical} \}$ 

can be shown that:

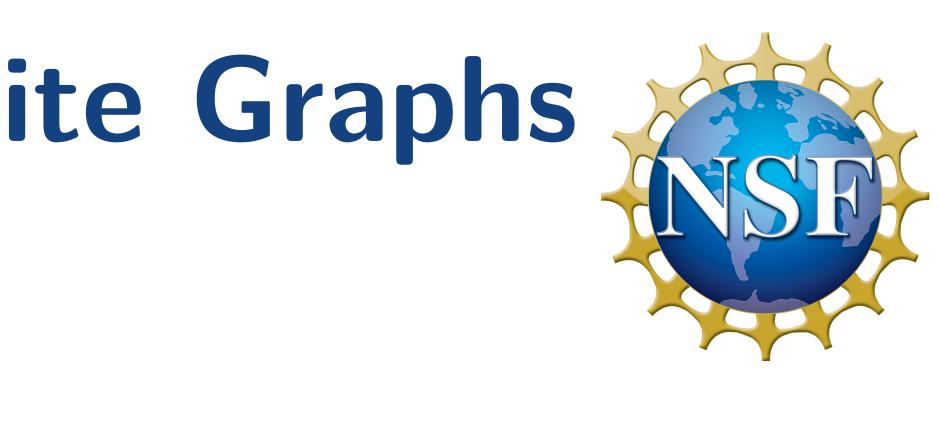
$$\sum_{\substack{Y \subseteq [n-1] \\ |J|=j}} \left| \bigcap_{i \in J} L_i \right| = (1+n-j)^m (n-j)! S(n,n-j)$$

The exceedance set of a permutation  $\sigma \in S_{m+n}$ is  $ex(\sigma) = \{i \mid \sigma(i) > i\}$ . Then, letting  $T(m,n) = \{ \sigma \in S_{m+n} \mid ex(\sigma) = \{1,\ldots,m\} \}$ we give a simple bijection between  $\mathcal{A}(K_{m+1,n},q)$ (with q in the (m + 1)-set) and T(m, n):

# **Conclusions/Future Questions**

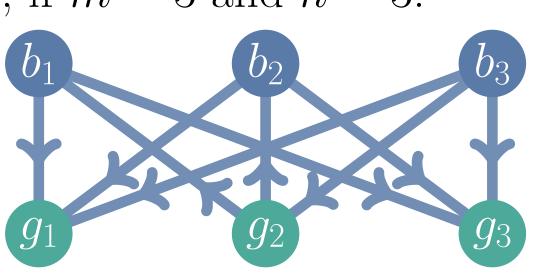
Our formulas gives ways to quickly compute the number of AOs for complete multipartite graphs. The idea of first considering a partially unlabeled graph might be useful in other problems.

[1] P. Cameron, C. Glass, and R. Schumacher. Acyclic orientations and poly-bernoulli numbers. *arXiv preprint arXiv:1412.3685*, 2014. [2] T. Helmuth, W. Perkins, and G. Regts. Algorithmic pirogov-sinai theory. arXiv preprint arXiv:1806.11548, 2018. [3] R. Kenyon and P. Winkler. Branched polymers. The American Mathematical Monthly, 116(7):612–628, 2009.



#### **AOs and Permutations**

- **1** Remove q, giving the set of AOs of  $K_{m,n}$  with no sink in the m-set.
- 2 Effectively equate (canonical) topological sorts with (canonical) cycle decompositions.
- For example, if m = 3 and n = 3:



gives the topological sort/cycle decomposition  $b_3g_2b_1b_2g_3g_1 \quad \longleftrightarrow \quad 351264 \quad \longleftrightarrow \quad (35)(1264)$ 

• Which G maximize  $|\mathcal{A}(G)|$  for fixed |V|, |E|? [1] • Similar bijection for complete multipartite case? • Count #AOs for grid graphs and hypercubes?

#### References