

Background

It is a classical result that any knot or link in \mathbb{R}^3 bounds a surface embedded in \mathbb{R}^3 . It turns out that knots and links also bound surfaces embeddeded in \mathbb{R}^4 . An active area of research explores knots and links that bound surfaces with simple topology.

Definition (Slice). A knot $K \subset S^3$ is *slice* if K bounds a smoothly properly embedded disk $D \subset B^4$.

Figure 1 shows a knot that bounds a disk in \mathbb{R}^3 ; however, the disk is not embedded since it intersects itself. After pushing the disk into \mathbb{R}^4 , it no longer intersects itself. Thus the knot is slice.



Fig. 2: Example χ -slice link



Fig. 1: Example surface bounded by a slice knot

The notion of χ -sliceness is a generalization of sliceness for links.

Definition (χ **-slice).** A link $L \subseteq S^3$ is χ -slice if L bounds a smoothly properly embedded surface $F \subseteq B^4$ with Euler characteristic 1 and no closed components.

In order to understand which links are χ -slice, one must apply algebraic tools (called obstructions) that can inform us that particular links are not χ -slice. It turns out that integer sublattices of \mathbb{Z}^n serve as an obstruction.



CUBIQUITOUS LATTICES AND χ -SLICENESS

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Motivation

One can associate a 3-manifold $\Sigma(L)$ to any link L. If L is an alternating nonsplit χ -slice link, then $\Sigma(L)$ is the boundary of a simple 4-manifold called a *rational 4-ball*.

Theorem ([1], [2]). Given an alternating nonsplit link L, if $\Sigma(L)$ bounds a rational 4-ball, then there exists an associated cubiquitous sublattice $\Lambda(L)$ of \mathbb{Z}^n .

Greene-Owens Conjecture. Given an alternating nonsplit link L, if there exists an associated cubiquous sublattice $\Lambda(L)$ of \mathbb{Z}^n , then $\Sigma(L)$ bounds a rational 4-ball.

Questions:

What conditions ensure or obstruct cubiquity? Is the Greene-Owens Conjecture true for torus links?

Cubiquity Results

Lemma 1. Let $\Lambda \subset \mathbb{Z}^n$ be a cubiquitous lattice. Then there exists a basis B for Λ such that each $b \in B$ satisfies

 $||b||^2 = b_1^2 + \dots + b_n^2 \le n + 3$

Lemma 2. Let $\Lambda \subset \mathbb{Z}^n$ be a cubiquitous lattice. Then there exists a basis B with entries in $\{-2, -1, 0, 1, 2\}$ such that each $b \in B$ has at most one entry with absolute value 2.



Fig. 6: Intuition for Lemma 1

Proposition. Let Λ be an orthogonal sublattice with orthogonal basis B. Denote the Wu element by $W = (z_1, \ldots, z_n) \in \mathbb{Z}^n$ and let O be the number of odd entries of W. If $\sum_{i=1}^{n} z_i^2 > 4n - 3O$, then Λ is not cubiquitous.

Definition. A sublattice $\Lambda \subset \mathbb{Z}^n$ is called *orthogonal* if it admits an orthogonal basis.

Theorem. Let Λ be orthogonal. Then Λ is cubiquitous if and only if it admits an orthogonal basis $\mathcal{B} = \{v_1, ..., v_n\}$ such that the matrix

$$B = \begin{bmatrix} | \\ v_1 & \dots \\ | \end{bmatrix}$$

is a block diagonal matrix and has blocks of the form [1], [2], or up to reordering and negating the standard orthonormal basis vectors of \mathbb{Z}^n .

Fig. 7: 3D Intuition for Lemma 2

$$v_n$$

Applications



Fig. 8: An example of a connected sum of torus links. Each k_i in the box indicates the number of half twists.

Methods

Next Steps

Currently, we are exploring the cubiquity of sublattices with *mediocre* bases, which include othogonal bases.

Definition (Mediocre Subset). A set $S = \{v_1, ..., v_n\}$ of linearly independent vectors in \mathbb{Z}^n is *mediocre* if

A mediocre sublattice of \mathbb{Z}^n is a sublattice a mediocre basis.

These subsets share many properties with *good subsets* that were explored by Lisca [3]. We hope to generalize our results on orthogonal lattices to mediocre lattices.

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References

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Our theorem has the following implication on connect sums of torus links.

Corollary. The Greene-Owens Conjecture is true for connected sums of alternating positive torus links. Moreover, a connected sum of alternating positive torus links is χ -slice if and only if the summands are of the form T(2, 1), T(2, 4), and T(2, 2) # T(2, 2).

To understand cubiquity in higher dimensions, we utilized the result from Lemma 2 to make a computationally reasonable set of orthogonal bases to check against the cubiquity obstruction given in the Proposition. This mainly entailed two algorithms:

an orthongal basis generator and a Wu obstruction checker. The code and documentation are available at https://github.com/ericaychoi/cubiquity-check.

$$_{i}, v_{j} \rangle = \begin{cases} 0 & |i - j| \ge 1 \\ 0, -1 & |i - j| = 1 \end{cases}$$

[1] Joshua Greene and Stanislav Jabuka. "The slice-ribbon conjecture for 3-stranded pretzel links". In: American Journal [2] Joshua Greene and Brendan Owens. "Alternating links, rational balls, and cube tilings". In: arXiv:2212.06248

[3] Paolo Lisca. "Lens spaces, rational balls and the ribbon conjecture". In: Geometry & Topology 11 (2007), pp. 429–472.

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