SGD - Rate of convergence to Gaussian using Steins method

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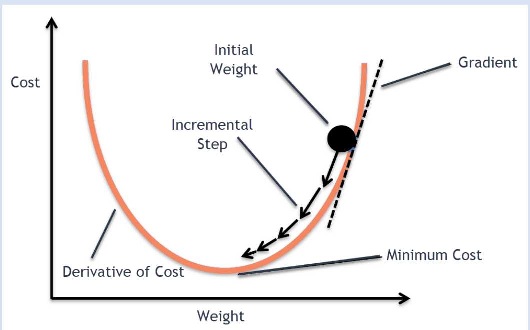
Illuminative example $f(x) = x^2/2$:

- **Stochastic Gradient Descent:**

$$X_{k+1}^{(\alpha)} = X_k^{(\alpha)} + \alpha(-\nabla f(X_k^{(\alpha)}) + w_k)$$

- **Scaled Iterate**

$$Y_k^{(\alpha)} = \frac{X_k^{(\alpha)} - x^*}{\sqrt{\alpha}}$$



- **Stochastic Gradient Descent:**

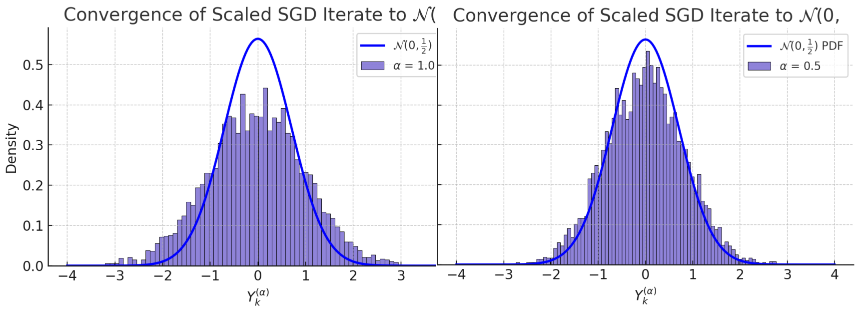
$$X_{k+1}^{(\alpha)} = (1-\alpha)X_k^{(\alpha)} + \alpha w_k$$

- **Scaled Iterate:**

$$Y_k^{(\alpha)} = \frac{X_k^{(\alpha)}}{\sqrt{\alpha}}, Y_k^{(\alpha)} \to \mathcal{N}(0, \frac{1}{2-a}) \quad [1]$$

- **Updated SGD:**

$$Y_k^{(\alpha)} = (1-\alpha)^k Y_0^{(\alpha)} + \sum_{i=0}^{k-1} (1-\alpha)^{k-1-i}\sqrt{\alpha}\, w_i.$$



Results:

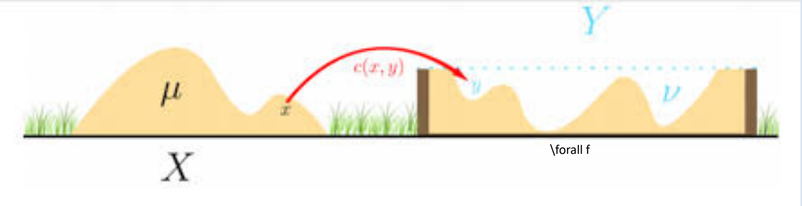
$$d_W(W,Z) = \lim_{k\to\infty} d_W(W_k, Z) = O(\sqrt{\alpha})$$

General 1 Dimensions $f: \mathbb{R} \to \mathbb{R}$

Wasserstein Distance:

$d_W(W,Z) = \sup_{h\in H} |\mathbb{E}[h(W)] - \mathbb{E}[h(Z)]|, \quad H = \{h : \mathbb{R} \to \mathbb{R} : |h(x) - h(y) \leq |x-y|\}$

A way to think about the is the optimal transport problem:



Assumptions:

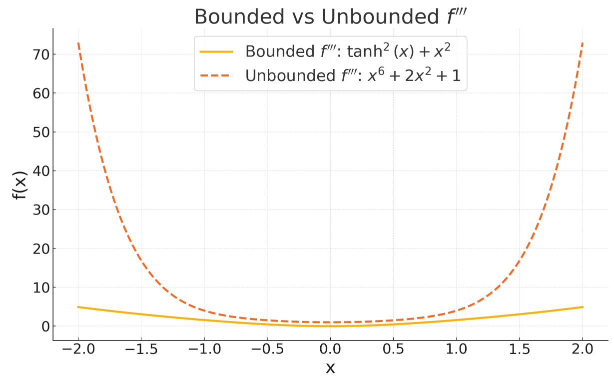
1. f(x) is thrice differtiable such that $\boxed{||f'''(x)||_\infty = C < \infty}$
2. L-smooth
3. σ-convex
4. $\mathbb{E}[w_k] = 0, \mathrm{Var}[w_k] = \Sigma$

Theorem:

1. There exists $a' > 0$, $\forall \alpha \in (0, a')$, then $Y_k^{(\alpha)} \xrightarrow{d} Y^{(\alpha)}$
2. For $\{\alpha_k\} \in (0, a')$, $\lim_{k\to\infty} \alpha_k = 0$, then

$$Y^{(\alpha)} \xrightarrow{d} Y \sim \mathcal{N}(0, \Sigma_y), \quad \Sigma_y = \frac{\Sigma}{2f''(x^*)}$$

3. $d_W(Y^{(\alpha)}, Y) \leq L_1\sqrt{\alpha} + L_2\alpha = \boxed{O(\sqrt{\alpha})}$



Proof for $f(x) = x^2/2$

Steins Method:

$$\forall f, \quad \mathbb{E}[A f(X)] = 0 \leftrightarrow X \sim Z$$

$$d_W(W,Z) = \sup_{h\in H} |\mathbb{E}[A f_h(W)]|$$

Definitions:

$$W_k = \frac{Y_k^{(\alpha)} - \mathbb{E}[Y_k^{(\alpha)}]}{\sqrt{\mathrm{Var}(Y_k^{(\alpha)})}}$$

$$W_k' = W_k - \frac{1}{\sigma}(1-\alpha)^{k-1-i}\sqrt{\alpha}\, w_i + \frac{1}{\sigma}(1-\alpha)^{k-1-i}\sqrt{\alpha}\, w_i'$$

$$\sigma^2 = \mathrm{Var}[Y_k^{(\alpha)}]$$

Exchangeable pairs: $(W, W') \overset{d}{=} (W', W)$

a-Stein pair: $E[W \mid W'] = (1-a)W$

Proposition 1 [2]:
Suppose (W,W') is a-Stein pair, $\mathbb{E}[W^2] = 1$, Z is standard normal, then

$$d_w(W,Z) \leq \frac{\sqrt{\mathrm{Var}(\mathbb{E}[(W'-W)^2 \mid W}}{\sqrt{2\pi a}} + \frac{\mathbb{E}|W'-W|^3}{3a}$$

Proof Sketch

1. (W_k, W_k') form a (1/n)-Stein pair:
2. Proposition 1: $\lim_{k\to\infty} d_W(W_k, Z) = L_1\sqrt{\alpha}, \quad L_1 > 0$

References:

[1] Zaiwei Chen, Shancong Mou, and Siva Theja Maguluri. Stationary behavior of constant stepsize sgd type algorithms: An asymptotic characterization. Proc. ACM Meas. Anal. Comput. Syst., 6(1), February 2022.

[2] Nathan Ross. Fundamentals of stein's method. Probability Surveys, 8:210–293, 2011.