# Upper Bound on Number of Pseudo-Equivalence Classes of Simple Line Arrangements in the Real Projective Plane

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## INTRODUCTION

An open question in discrete and combinatorial geometry is the enumeration of simple straight lines in the plane: that is, given n lines, how many different arrangements exist? Although the usual definition of equivalence is defined as a mapping existing between the respective vertices, edges, and faces preserving incidences, we define two line arrangements as pseudo-equivalent if their cell complexes contain the same number of k-gons for any k. We restrict our study to simple line arrangements in the projective plane, and we employ the use of generating functions to find an upper bound on the number of pseudo-equivalent arrangements. We also prove a property of such line arrangements that allow us to tighten our upper bound.

## EXAMPLES OF ARRANGEMENTS

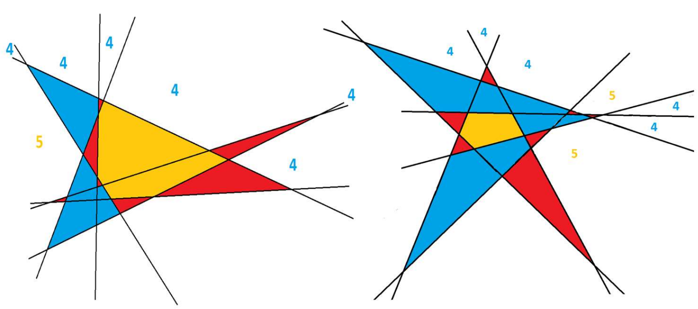


*Figure 1: Two simple pseudo-equivalent line arrangements of 7 lines in the real projective plane.*

The two arrangements above, each of n = 7, are not equivalent. This can be seen as two pentagons (yellow) share an edge in the left arrangement, but not in the right arrangement. However, the two arrangements are considered pseudo-equivalent as each have three 5-gons, twelve 4-gons, and seven 3-gons.

## METHOD

1. For n simple lines in $\mathbb{P}(\mathbb{R}^2)$ we find for the number of edges (E) and number of faces (F):
$$E = n^2 - n.$$
$$F = \binom{n-1}{2} + n$$
2. Let $P_i$ be the number of k-gons where k = i. Then we have the following system of equations:
$$\left\{ \begin{array}{c} 3P_3 + 4P_4 + \ldots + nP_n = 2(n^2 - n) \\ P_3 + P_4 + \ldots + P_n = \binom{n-1}{2} + n \end{array} \right\}$$
where $P_3, P_4, \ldots, P_n \in \mathbb{N}$.
3. Subtracting three times the lower equation from the first eliminates $P_3$ and yields:
$$\left\{ P_4 + 2P_5 + \ldots + (n-3)P_n = \tfrac{1}{2}(n^2 - n - 6) \right\}$$
4. We use the method of generating functions to find the number of integer solutions. The generating function for the equation above is given by:
$$\prod_{i=1}^{n-3} \frac{1}{1-x^i}$$
5. Using a computer algebra system, we find the coefficient on the $x^{\frac{1}{2}(n^2-n-6)}$ term for a given n. This is the upper bound.
6. With the lemma proven at right, we can modify the system of equations to substitute 0 or 1 for $P_n$ and using the same procedure of generating functions as above, we achieve a tighter upper bound.
7. Future study of linear Diophantine inequalities as well as the known bounds of $P_3$ can improve the bound further still.

## LEMMA

Let $\mathbb{A}$ be a simple line arrangement of $n$ lines in $\mathbb{P}(\mathbb{R}^2)$. Let $P_i$ denote the number of i-sided faces of $\mathbb{A}$. Then one of the following is true

i. $P_n = 1$ and $P_{n-1} = 0$

ii. $P_n = 0$.

## RESULTS

The chart below provides a summary for the upper bounds of the number of pseudo-equivalent line arrangements of n lines in the projective plane. Two upper bounds are presented: one without the restriction imposed by the lemma above and one with the restriction imposed by the lemma above.

*Table 1: Actual and upper bounds on the number of simple pseudo-equivalent line arrangements in the real projective plane.*

| n | Actual | Upper Bound without Lemma | Upper Bound with Lemma |
|---|---|---|---|
| 3 | 1 | 1 | 1 |
| 4 | 1 | 1 | 1 |
| 5 | 1 | 4 | 1 |
| 6 | 4 | 19 | 8 |
| 7 | 9 | 84 | 45 |
| 8 | ? | 377 | 229 |

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