Math 4317, Assignment 2A

§1.3 Vector Spaces

1. Gunning §1.3 Group I Problem 1
2. Gunning §1.3 Group I Problem 6

§1.2 Groups, Rings, Fields

3. Show that $0a = 0$ for any element a in a ring R.
4. Let R be a ring and P a subset of R such that

$$R = -P \cup \{0\} \cup P = -P \coprod \{0\} \coprod P \qquad \text{is a partition}$$

where $-P = \{-a : a \in P\}$. Show that exactly one of the following holds for every $a \in R$:

   1. $a \in -P$,
   2. $a = 0$,
   3. $a \in P$.

5. Let R be a ring and P a subset of R such that for every $a \in R$ exactly one of the following holds:

   1. $a \in -P$,
   2. $a = 0$,
   3. $a \in P$

where $-P = \{-a : a \in P\}$. Show

$$R = -P \cup \{0\} \cup P = -P \coprod \{0\} \coprod P \qquad \text{is a partition}$$

6. Gunning §1.2 Group II Problem 9
7. Gunning §1.2 Group II Problem 10

§1.1-2 Sets and Numbers

8. Recall the bimonoid $\mathbb{N}_0 \times \mathbb{N}_0 = \{(a, b) : a, b \in \mathbb{N}_0\}$ with respect to the operations

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b)(c, d) = (ac + bd, ad + bc).$$

Show that

$$(a, b) \sim (c, d) \qquad \Longrightarrow \qquad a + d = b + c$$

defines an **equivalence relation** on $\mathbb{N}_0 \times \mathbb{N}_0$.

9. Let $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ denote the ring of integers and let $\mathbb{Z}^* = \{\pm 1, \pm 2, \pm 3, \ldots\}$ the nonzero elements. Define operations of addition and multiplication in $\mathbb{Z} \times \mathbb{Z}^*$ based on the addition and multiplication of fractions you know and the identification

$$(m, n) \in \mathbb{Z} \times \mathbb{Z}^* \quad \longleftrightarrow \quad \frac{m}{n} \in \mathbb{Q}.$$

What ring properties are satisfied by $\mathbb{Z} \times \mathbb{Z}^*$ under these operations?

10. If $\mathcal{C} = \{(a_\alpha, b_\alpha)\}_{\alpha \in \Gamma}$ is a collection of disjoint intervals in the real line $\mathbb{R}$, then Γ is (at most) countable.

Monotone Functions

11. Show that the sum of two monotone (non-decreasing) functions is monotone (non-decreasing).

Let $u : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function. Recall that

$$u_+(x) = \inf u((x, b)) \qquad \text{and} \qquad u_-(x) = \sup u((a, x))$$

are well-defined for each $x \in \mathbb{R}$ and u is continuous at $x \in \mathbb{R}$ if and only if the jump at x given by $S(x) = u_+(x) - u_-(x) = 0$.

12. If u and v are non-decreasing functions on $\mathbb{R}$ and x is a point of discontinuity for u, then x is a point of discontinuity for $u + v$.

13. If u and v are non-decreasing functions on $\mathbb{R}$ and x is a point of continuity for u **and** v, then x is a point of continuity for $u + v$.

14. Consider $u_n : \mathbb{R} \to \mathbb{R}$ by

$$u_n(x) = \begin{cases} -1/n^2, & x < 1/n \\ 1/n^2, & x \geq 1/n \end{cases} \qquad \text{for } n \in \mathbb{N}.$$

(a) Plot (draw the graph of)

$$f_k(x) = \sum_{n=1}^{k} u_n(x)$$

for $k = 1, 2, 3, 4$.

(b) Does

$$f(x) = \sum_{n=1}^{\infty} u_n(x)$$

make sense as a non-decreasing function? If so what is the set of discontinuities of f?