Math 4317, Assignment 2B

§1.3 Vector Spaces

1. (direct sum) Given two subspaces V_1 and V_2 of a vector space V, we set

$$V_1 + V_2 = \{v + w : v \in V_1 \text{ and } w \in V_2\}.$$

   This is called the **sum** of the subspaces V_1 and V_2. If $V_1 \cap V_2 = \{\mathbf{0}\}$, we say $V_1 + V_2$ is a **direct sum** and write $V_1 + V_2 = V_1 \oplus V_2$.

   (a) Show that $V_1 + V_2$ is a subspace of V.

   (b) Show that if $w \in V_1 \oplus V_2$, then there are **unique** vectors $x \in V_1$ and $y \in V_2$ such that $w = x + y$.

2. (direct product) Given any two vector spaces V_1 and V_2 over the same field, the **product space** $V_1 \times V_2$ is the Cartesian product with addition

$$(v, w) + (x, y) = (v + x, w + y)$$

   and scaling

$$a(v, w) = (av, aw).$$

   Show that $V_1 \times V_2 = W_1 \oplus W_2$ for appropriate subspaces W_1 and W_2 which are isomorphic to V_1 and V_2 respectively.

3. (span) Given any subset S of a vector space V, the **span** of S is defined to be

$$\left\{ \sum_{j=1}^{k} a_j v_j : a_1, \ldots, a_k \text{ are scalars in the field, and } v_1, \ldots, v_k \in V \right\}.$$

   That is, the span of S is the set of all **finite linear combinations** of vectors in S. The span of S is denoted $\operatorname{span}(S)$.

   (a) Show $\operatorname{span}(S)$ is a subspace of V.

   (b) If V_1 and V_2 are subspaces of a vector space V and $V_1 \cap V_2 = \{\mathbf{0}\}$ with $\{v_1, \ldots, v_k\}$ a basis for V_1 and $\{w_1, \ldots, w_\ell\}$ a basis for V_2, then show

$$\{v_1, \ldots, v_k\} \cup \{w_1, \ldots, w_\ell\}$$

   is a basis for $V_1 \oplus V_2$. Note: The problem above has been corrected. It previously read "...then show

$$\{v_i + w_j : i = 1, \ldots, k \text{ and } j = 1, \ldots, \ell\}$$

   is a basis..."

4. Gunning §1.3 Group I Problem 3

5. The **center** C of a ring R is the set of all elements which commute with all others with respect to multiplication:

$$C = \{a \in R : ax = xa \text{ for all } x \in R\}.$$

What is the center of the ring of all $n \times n$ matrices under matrix multiplication?

Ordered rings

Problems 6,7,8, and 9 refer to the situation in which $P \subset R$ is a set of positives determining an order in a ring R with

$$a < b \qquad \Longleftrightarrow \qquad b - a \in P.$$

6. Show that $\mathbb{Z}_3$ is a field, but $\mathbb{Z}_3$ is not an ordered field in the sense of being ordered by a set of positives. Is it possible to have an order relation on $\mathbb{Z}_3$?
7. Show the following:
   (a) $\mathcal{T} = \{(a, b) : b - a \in P \cup \{0\}\}$ is a total order on R.
   (b) $P = \{x \in R : x > 0\}$.
8. Show the following:
   (a) $a \in R\backslash\{0\}$ implies $a^2 \in P$.
   (b) $1 > 0$.
   (c) If $a \leq b$ and $c > 0$, then $ac \leq bc$, but if $a < b$ and $c < 0$, then $ac > bc$.
   (d) If $a \leq b$, then $-a \geq -b$.
9. An element in a ring is a **zero divisor** if $a \in R\backslash\{0\}$ and there is some $b \in R\backslash\{0\}$ for which $ab = 0$. An **integral domain** is a ring with no zero divisors.
   (a) Give an example of a ring which is not an integral domain.
   (b) Give an example of a ring which is an integral domain but not a field.
   (c) Show that every ordered ring (ordered by a set of positives) is an integral domain.
10. Gunning §1.2 Group II Problem 8

11. (integers) $\nu_0 : \mathbb{N}_0 \to \mathbb{Z}$ by $\nu_0(n) = [(n, 0)]$ is order preserving. Hint: Recall that $\mathbb{N}_0$ is ordered by set inclusion so that

$$1 = \{0\} \subset \{0, 1\} = 2,$$

and $\mathbb{Z} = \{[(n, m)] : n, m \in \mathbb{N}\}$ is ordered by the set of positives $\nu_0(\mathbb{N})$.

12. (rationals) Show that every nonzero rational number $[(p, q)]$ with $(p, q) \in \mathbb{Z} \times \mathbb{Z}^*$ can be written uniquely in one of the following two forms

$$[(p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}, q_1^{\ell_1} q_2^{\ell_2} \cdots q_m^{\ell_m})]$$

where $p_1 < p_2 < \cdots p_n$ and $q_1 < q_2 < \cdots < q_m$ are prime natural numbers with $\{p_1, \ldots, p_n\} \cap \{q_1, \ldots, q_m\} = \phi$, or

$$[(-p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}, q_1^{\ell_1} q_2^{\ell_2} \cdots q_m^{\ell_m})]$$

where $p_1 < p_2 < \cdots p_n$ and $q_1 < q_2 < \cdots < q_m$ are prime natural numbers with $\{p_1, \ldots, p_n\} \cap \{q_1, \ldots, q_m\} = \phi$.

The assertion of uniqueness here requires some care. The exponents $k_1, \ldots, k_n, \ell_1, \ldots, \ell_m$ are non-negative integers. We must allow zero exponents to obtain 1 as in, for example, the integer classes $[(n, 1)]$ and also the fractions $[(1, n)]$. This possibility (of zero exponents) implies a certain non-uniqueness for the sets of primes $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_m\}$. This is because, subject to the requirement that $\{p_1, \ldots, p_n\} \cap \{q_1, \ldots, q_m\} = \phi$, any additional distinct primes p_* and q_* may be included with exponents $k_* = \ell_* = 0$. In all cases, however, the products $p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$ and $q_1^{\ell_1} q_2^{\ell_2} \cdots q_m^{\ell_m}$ are unique. If one of the products $p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$ or $q_1^{\ell_1} q_2^{\ell_2} \cdots q_m^{\ell_m}$ is 1, then for that product there is no further uniqueness to discuss. If, however, one of the sets of exponents $\{k_j : k_j \neq 0\}$ or $\{\ell_j : \ell_j \neq 0\}$ is nonempty, then the corresponding set of primes $\{p_j : k_j \neq 0\}$ or $\{q_j : \ell_j \neq 0\}$ is unique.

Monotone Functions and sequences

A **sequence** is a function from $\mathbb{N}$ or $\mathbb{N}_0$ to a set. Here, as with our consideration of monotone functions, we will consider sequences taking values among the real numbers. For these particular functions we use a special/unusual notation: Instead of writing $f : \mathbb{N} \to \mathbb{R}$ or $f(n)$ for the image of $n \in \mathbb{N}$, we use a subscript and write a_n as the value assigned to $n \in \mathbb{N}$. For the whole sequence/function we write

$$\{a_n\}_{n=1}^{\infty} \qquad \text{or sometimes} \qquad a_1, a_2, a_3, \ldots.$$

As the notation suggests, we will also consider the **values of the function** (also called the sequence) as a subset of the real numbers:

$$\{a_n : n = 1, 2, 3, \ldots\}.$$

No confusion should result from this slight abuse of notation.

A sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is said to be **monotone non-decreasing** if $a_{n+1} \geq a_n$ for $n = 1, 2, 3, \ldots$. There are two possibilities for a monotone non-decreasing sequence: Either the sequence is bounded above, or it is not bounded above, i.e., either the set of sequence values is bounded above, or it is not. In the first case, there is a **least upper bound** and we write

$$\lim_{n\to\infty} a_n = \sup\{a_n : n = 1, 2, 3, \ldots\} \qquad \text{or} \qquad \lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} a_n.$$

If the sequence is not bounded above, then we write

$$\lim_{n\to\infty} a_n = \sup\{a_n : n = 1, 2, 3, \ldots\} = \sup_{n\in\mathbb{N}} a_n = \infty$$

and we say the **limit exists in the extended real numbers**. The **extended real numbers** often denote the set $\mathbb{R} \cup \{\infty\}$ and one also uses interval notation so that

$$\mathbb{R} \cup \{\infty\} = (-\infty, \infty]$$

and other intervals $[a, \infty]$ are also possible. Note that the extended real numbers are quite different from the second infinite ordinal $\omega + 1 = \{0, 1, 2, \ldots, \omega\}$. They are also somewhat different from the second uncountable ordinal $\Omega + 1 = \Omega \cup \{\Omega\}$. The symbol ∞ is different from ω and from Ω. The arithmetic associated with it is different. Sometimes the **extended real numbers** include two additional symbols, ∞ and $-\infty$, so that we have an interval $[-\infty, \infty]$. In this case, one does not mean that $-\infty$ is the additive inverse of ∞.

13. What is the difference between a least upper bound and a supremum?

14. Let $a_1, a_2, a_3, \ldots$ be a sequence of non-negative real numbers.

(a) Show the sequence $\{s_k\}_{k=1}^{\infty}$ defined by

$$s_k = \sum_{n=1}^{k} a_n$$

is a monotone non-decreasing sequence. Thus, the limit of $\{s_k\}_{k=1}^{\infty}$ always exists in the extended real numbers, and we write

$$\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} s_k.$$

(b) Show the following: If $\sum_{n=1}^{\infty} a_n = s \in \mathbb{R}$, then for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ for which

$$k > N \qquad \Longrightarrow \qquad s - \epsilon < s_k < s + \epsilon.$$

(c) Show the following: If $\sum_{n=1}^{\infty} a_n = \infty$, then for any $M > 0$, there is some $N \in \mathbb{N}$ for which

$$k > N \qquad \Longrightarrow \qquad s_k > M.$$

In the bounded case one says the **series** $\sum_{n=1}^{\infty} a_n$ **converges** to the sum s. In the unbounded case, sometimes one says the series converges to ∞. In this latter case, it is also said that the series **diverges** to ∞. These phrases mean the same thing.

15. Let $a_1, a_2, a_3, \ldots$ be **any** sequence of real numbers (not necessarily monotone). Again, we consider the sequence $\{s_k\}_{k=1}^{\infty}$ of **partial sums** defined by

$$s_k = \sum_{n=1}^{k} a_n.$$

If there is a real number $s \in \mathbb{R}$ such that

for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ for which

$$k > N \qquad \Longrightarrow \qquad s - \epsilon < s_k < s + \epsilon,$$

then we write

$$\lim_{k \to \infty} s_k = s \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n = s,$$

and say the series **converges** to the sum s. Show that if $\sum_{n=1}^{\infty} |a_n| \in \mathbb{R}$, then $\sum_{n=1}^{\infty} a_n \in \mathbb{R}$. In this case, we say the series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent**.