**Math 4317, Assignment 3A**

§1.3 Vector Spaces

I've started to include some problems on vector spaces related to Gunning's long section 1.3, so perhaps I should provide some introductory comments, which I will do here. Gunning has provided what I think is a rather wonderful presentation of linear algebra, especially when considered as a review. On the one hand, the material is important, and this is perhaps a good opportunity for you to review linear algebra. Also, there are a few topics/concepts which you almost certainly do not know and probably many more which you did not learn well, and this could be an opportunity to pick up that material—it probably should be. Thirdly, linear algebra provides a pretty good opportunity to learn how to write rigorous proofs without some of the technicalities of analysis proper. On the other hand, I think most of you have been exposed to a fair amount of linear algebra, the section is very long, most authors of analysis texts give a much more limited treatment (Buck puts linear algebra in an appendix for example), and we could easily spend a semester looking carefully at the material included by Gunning and never get to the important material on real functions and function spaces: convergence, continuity, and differentiability, etc.. That would be unacceptable to me, so I propose the following: You read the linear algebra section slowly on your own over the course of the semester, and we will go on to Chapter 2 with regard to the homework assignments and discussion in class, except for the inclusion of one or two problems on each semi-assignment on linear algebra.

I have nominally started this strategy in the previous two assignments, but this assignment will explicitly include material from Chapter 2, so I make this announcement/declaration here. And we'll see how it goes. If there's something important from linear algebra on which we need to focus, we can revise this strategy and do that.

1. Prove the **basis theorem** (Gunning Chapter 1, §1.3, Theorem 1.7).

§1.2 Groups, Rings, Fields

Note: The following two problems were inadvertently repeated. They appear already as Problems 6 and 7 of Assignment 2A. As an administrative/reorganization comment for future use of these problem sets: Assignment 2B covered three rather lengthy modules (operator norm on normed vector spaces, inner product spaces, and epsilon-delta continuity). As a consequence there were no problems directly involving monotone functions. There were also two arguably isolated problems concerning operators and functionals under the heading of Groups, Rings, and Fields. Thus, these two problems (Problems 4 and 5 on Assignment 3B) could be moved here to replace these two repeat problems and leaving room for two problems on monotone functions on Assignment 3B.

2. Gunning §1.2 Group II Problem 9

3. Gunning §1.2 Group II Problem 10

§1.1-2 Sets and Numbers

For the next three problems below, let $X$ be a partially ordered set. Recall that the basic relation on $X$ is denoted by "$\leq$" and we write

$$a < b \qquad \text{to mean } a \leq b \text{ but } a \neq b.$$

**Definition 1** *A subset $A$ in the partially ordered set $X$ is* **bounded above** *if there is an element $x \in X$ for which*

$$a \leq x \qquad \textit{for all } a \in A.$$

The set of all upper bounds $B$ for a set $A$ in a given set $X$, that is

$$B = \{x \in X : a \leq x \text{ for all } a \in A.\},$$

may be empty. If $A$ is bounded above, it may be that $B \cap A = \phi$. If $B \cap A \neq \phi$, then $B \cap A = \{a_{\max}\}$ is a singleton, and we call the element $a_{\max}$ the **maximum** of $A$ and write

$$a_{\max} = \max A.$$

4. Give examples of the following:
   (a) A partially ordered set $X$ with a subset $A$ in which the set
   $$B = \{x \in X : a \leq x \text{ for all } a \in A.\} = \phi$$
   (b) A partially ordered set $X$ with a subset $A$ in which the set
   $$B = \{x \in X : a \leq x \text{ for all } a \in A.\} \neq \phi,$$
   i.e., $A$ is bounded above, and $B \cap A = \phi$.
   (c) A partially ordered set $X$ with a subset $A$ which is bounded above and the set of upper bounds $B$ satisfies $B \cap A \neq \phi$.

   Show the following:
   (d) If $X$ is a partially ordered set with $A \subset X$ and $A$ is bounded above with the set of upper bounds $B$ satisfying $B \cap A \neq \phi$, then $B \cap A$ is a singleton.
   (e) If $A \subset \mathbb{Z}$ is bounded above, then show $\max A$ is well-defined.

**Definition 2** *A subset $A$ in a partially ordered ordered set $X$ is* **bounded below** *if there is an element $x \in X$ for which*

$$x \leq a \qquad \textit{for all } a \in A.$$

If the set of all lower bounds $L$ for a set $A$ in a given set $X$, that is

$$L = \{x \in X : x \leq a \text{ for all } a \in A.\},$$

satisfies $L \cap A = \{a_{\min}\}$ is a singleton, we call the element $a_{\min}$ the **minimum** of $A$ and write

$$a_{\min} = \min A.$$

5. Show the following:
   (a) If $A$ is bounded above in $X$, then the set of upper bounds

   $$B = \{x \in X : a \leq x \text{ for all } a \in A.\} \neq \phi$$

   is bounded below.
   (b) In the situation of the previous part, show that if $\max A$ is well-defined, then $\min B$ is well-defined and $\max A = \min B$.

   Give examples of the following:
   (c) A partially ordered set $X$ with a subset $A$ which is bounded above and for which $\max A$ is **not** well-defined but the set of upper bounds $B$ does have $\min B$ well-defined.
   (d) A partially ordered set $X$ with a subset $A$ which is bounded above and for which **neither** $\max A$ **nor** $\min B$ is well-defined where $B$ is the set of upper bounds for $A$.

**Definition 3** *If $X$ is a partially ordered set and $A \subset X$ is bounded above, if* $\min B$ *where $B$ is the set of upper bounds of $A$ is well-defined, we write*

$$\sup A = \min B,$$

*and* $\min B$ *is called the* **supremum** *of $A$ or* **least upper bound**.

**Definition 4** *A partially ordered set $X$ in which every bounded set has a least upper bound is said to have the* **least upper bound property** *or to be* **complete**.

6. (rationals) Show that the rational numbers $\mathbb{Q}$ are not complete.

§2.1 Normed Vector Spaces

A **norm** on a real vector space $V$ is a function $\| \cdot \| : V \to [0, \infty)$ satisfying the following:

**(i positive definite)** $\|v\| = 0$ if and only if $v = \mathbf{0}$.

**(ii non-negative homogeneous)** $\|cv\| = |c|\|v\|$ for all $c \in \mathbb{R}$ and $v \in V$.

**(iii triangle inequality for norms)** $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

7. If $\| \cdot \| : \mathbb{R} \to [0, \infty)$ is a norm on $\mathbb{R}^1$ (as a vector space over the real field $\mathbb{R}$), then there exists a positive constant $c \in \mathbb{R}$ such that $\|\alpha\| = c|\alpha|$ where

$$|\alpha| = \begin{cases} \alpha & \text{if } \alpha \geq 0 \\ -\alpha & \text{if } \alpha < 0. \end{cases}$$

8. Show the $\ell^1$ norm
$$\|\mathbf{x}\| = \sum_{j=1}^{n} |x_j|$$
is a norm on $\mathbb{R}^n$.

9. Show the $\ell^2$ norm
$$\|\mathbf{x}\| = \sqrt{\sum_{j=1}^{n} x_j^2}$$
is a norm on $\mathbb{R}^n$. This norm is called the **Euclidean norm**.

10. Show the $\ell^\infty$ norm
$$\|\mathbf{x}\| = \max\{|x_j| : 1 \leq j \leq n\}$$
is a norm on $\mathbb{R}^n$. This is called the **supremum norm** (or the **sup norm**).

11. Gunning §2.1 Group I Problem 1

Equivalent Norms

Two norms $\|\ \cdot\ \|_A$ and $\|\ \cdot\ \|_B$ on the same vector space $V$ are **equivalent** if there are real numbers $\alpha$ and $\beta$ such that

$$\|v\|_A \leq \alpha\|v\|_B \qquad \text{and} \qquad \|v\|_B \leq \beta\|v\|_A \quad \text{for all } v \in V.$$

12. (a) Show the equivalence of norms is an equivalence relation on the set of norms.
    (b) Show the $\ell^1$, $\ell^2$, and $\ell^\infty$ norms on $\mathbb{R}^n$ are all equivalent and give explicit best constants (for the inequalities).

Monotone Functions and sequences

A **sequence** is a function from $\mathbb{N}$ or $\mathbb{N}_0$ to a set. Here, as with our consideration of monotone functions, we will consider sequences taking values among the real numbers. For these particular functions we use a special/unusual notation: Instead of writing $f : \mathbb{N} \to \mathbb{R}$ or $f(n)$ for the image of $n \in \mathbb{N}$, we use a subscript and write $a_n$ as the value assigned to $n \in \mathbb{N}$. For the whole sequence/function we write

$$\{a_n\}_{n=1}^{\infty} \qquad \text{or sometimes} \qquad a_1, a_2, a_3, \ldots.$$

As the notation suggests, we will also consider the **values of the function** (also called the sequence) as a subset of the real numbers:

$$\{a_n : n = 1, 2, 3, \ldots\}.$$

No confusion should result from this slight abuse of notation.

A sequence of real numbers $\{a_n\}_{n=1}^\infty$ is said to be **monotone non-decreasing** if $a_{n+1} \le a_n$ for $n = 1, 2, 3, \ldots$. There are two possibilities for a monotone non-decreasing sequence: Either the sequence is bounded above, or it is not bounded above, i.e., either the set of sequence values is bounded above, or it is not. In the first case, there is a **least upper bound** and we write

$$\lim_{n\to\infty} a_n = \sup\{a_n : n = 1, 2, 3, \ldots\} \qquad \text{or} \qquad \lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} a_n.$$

If the sequence is not bounded above, then we write

$$\lim_{n\to\infty} a_n = \sup\{a_n : n = 1, 2, 3, \ldots\} = \sup_{n\in\mathbb{N}} a_n = \infty$$

and we say the **limit exists in the extended real numbers**. The **extended real numbers** often denote the set $\mathbb{R} \cup \{\infty\}$ and one also uses interval notation so that

$$\mathbb{R} \cup \{\infty\} = (-\infty, \infty]$$

and other intervals $[a, \infty]$ are also possible. Note that the extended real numbers are quite different from the second infinite ordinal $\omega + 1 = \{0, 1, 2, \ldots, \omega\}$. They are also somewhat different from the second uncountable ordinal $\Omega{+}1 = \Omega{\cup}\{\Omega\}$. The symbol $\infty$ is different from $\omega$ and from $\Omega$. The arithmetic associated with it is different. Sometimes the **extended real numbers** include two additional symbols, $\infty$ and $-\infty$, so that we have an interval $[-\infty, \infty]$. In this case, one does not mean that $-\infty$ is the additive inverse of $\infty$.

13. What is the difference between a least upper bound and a supremum?

Convergence for Series of Non-negative Reals

14. Let $a_1, a_2, a_3, \ldots$ be a sequence of non-negative real numbers.
    (a) Show the sequence $\{s_k\}_{k=1}^\infty$ defined by

    $$s_k = \sum_{n=1}^k a_n$$

    is a monotone non-decreasing sequence. Thus, the limit of $\{s_k\}_{k=1}^\infty$ always exists in the extended real numbers, and we write

    $$\sum_{n=1}^\infty a_n = \lim_{k\to\infty} s_k.$$

    (b) Show the following: If $\sum_{n=1}^\infty a_n = s \in \mathbb{R}$, then for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ for which

    $$k > N \qquad \Longrightarrow \qquad s - \epsilon < s_k < s + \epsilon.$$

(c) Show the following: If $\sum_{n=1}^{\infty} a_n = \infty$, then for any $M > 0$, there is some $N \in \mathbb{N}$ for which

$$k > N \qquad \Longrightarrow \qquad s_k > M.$$

In the bounded case one says the **series** $\sum_{n=1}^{\infty} a_n$ **converges** to the sum $s$. In the unbounded case, sometimes one says the series converges to $\infty$. In this latter case, it is also said that the series **diverges** to $\infty$. These phrases mean the same thing.

Convergence for General Series of Reals

Let $a_1, a_2, a_3, \ldots$ be **any** sequence of real numbers (not necessarily monotone). Again, we consider the sequence $\{s_k\}_{k=1}^{\infty}$ of **partial sums** defined by

$$s_k = \sum_{n=1}^{k} a_n.$$

If there is a real number $s \in \mathbb{R}$ such that

for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ for which

$$k > N \qquad \Longrightarrow \qquad s - \epsilon < s_k < s + \epsilon,$$

then we write

$$\lim_{k\to\infty} s_k = s \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n = s.$$

In this case, we say the series **converges** to the sum $s \in \mathbb{R}$ or simply $\sum_{n=1}^{\infty} a_n \in \mathbb{R}$.

15. Show that if $\sum_{n=1}^{\infty} |a_n| \in \mathbb{R}$, then $\sum_{n=1}^{\infty} a_n \in \mathbb{R}$. In this case, we say the series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent**.